PHD

The application of Newton's method to simple bifurcation and turning point problems.

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Award date:
1979

Awarding institution:
University of Bath

THE APPLICATION OF NEWTON'S METHOD TO SIMPLE BIFURCATION AND TURNING POINT PROBLEMS.

submitted by G. MOORE for the

degree of Ph.D. of the University of Bath

1979

ABSTRACT

This thesis investigates the solution of equations of the form $G(\lambda, x) = 0$, $G: R \times X \rightarrow X$, where X is a real Banach space. Such equations are often called non-linear eigenvalue problems. If (λ^*, x^*) is a solution for which $G_x(\lambda^*, x^*)$ is invertible, there are well-known existence and uniqueness results for solutions near (λ^*, x^*), which are easily made constructive. However, in this thesis, we are interested in solutions (λ^*, x^*) for which $G_x(\lambda^*, x^*)$ is not invertible, and specifically in the case when $G_x(\lambda^*, x^*)$ has only a 1-dimensional null-space. Our approach is to apply the Newton-Kantorovich theorem, first to determine these so-called singular points, and second to compute nearby solutions. In the former case we modify the equations to avoid singular systems, and in the latter case we obtain accurate starting values which compensate for the near-singularity of $G_x(\lambda, x)$. Hence the advantages of a quadratically convergent method are retained.

Chapter 1 contains a brief, general introduction to non-linear eigenvalue problems in which we distinguish between the two important types of singular point, turning points and bifurcation points. In Chapter 2 we consider the common case $G(\lambda, 0) = 0$ for arbitrary λ. Thus $(\lambda, 0)$ is a solution, the singular points $(\lambda^*, 0)$ can be computed by standard methods, and the major difficulty is to determine solutions (λ, x), near $(\lambda^*, 0)$, with non-zero x-component. Turning points are the subject of Chapter 3, where we show how to compute both the points themselves and the solutions beyond them. In Chapter 4 we consider the problem of bifurcation points in a similar way and, finally, the stability of such points under small perturbations is discussed in Chapter 5.

ACKNOWLDEDGEMENT

I would like to thank my supervisor, Dr. Alastair Spence, for much encouragement and help during the preparation of this work, which was supported by a Science Research Council Research Studentship and a Bath University Research Studentship.

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CHAPTER 1

INTRODUCTION

In this thesis we are concerned with the solution of equations of the form

$$G(\lambda, x) = 0, \qquad (1.1)$$

where $\lambda \in R$, $x \in X$ (a real Banach space), and G is a continuously differentiable operator from R x X to X. If (λ°, x°) is a solution of (1.1) and $G_x(\lambda^\circ, x^\circ)$, the Frechet derivative of G with respect to x at (λ°, x°), possesses a bounded inverse on X, then the implicit function theorem guarantees the existence of a, b > o such that, for $|\lambda - \lambda^\circ| < a$, (1.1) has a solution $x(\lambda)$ unique in the ball $\|x - x^\circ\| < b$ and $x(\lambda)$ is continuously differentiable. As $x(\lambda^\circ) = x^\circ$, we have a unique smooth curve of solutions of (1.1), denoted by Γ, passing through (λ°, x°) as shown in Figure 1.1.

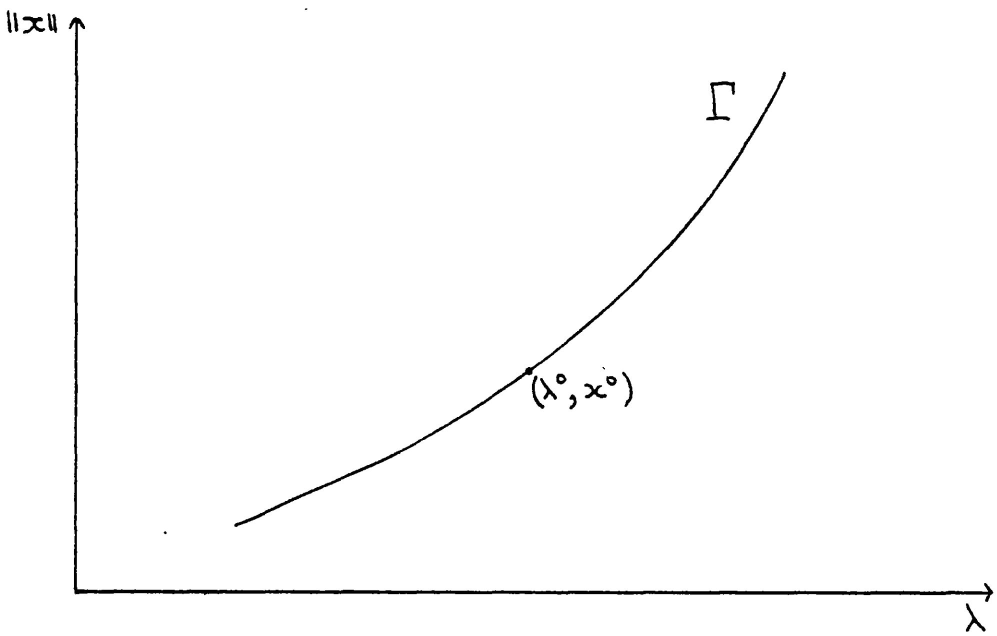

Fig. 1.1 Solution-Curve Γ through (λ°, x°).

In this thesis we are exclusively interested in solving (1.1) near solutions (λ^*, x^*) where $G_x(\lambda^*, x^*)$ is not invertible, and therefore the implicit function theorem cannot be applied. This means that there may well be no solution or more than one solution x "near" x^* of (1.1) for λ "near" λ^*. The term generally used to describe this behaviour is bifurcation theory, although different authors have slightly different definitions of the concept.

In many cases there is an explicitly known, smooth solution curve Γ with $(\lambda^*, x^*) \in \Gamma$ and $G_x(\lambda, x)$ invertible for $(\lambda, x) \in \Gamma - (\lambda^*, x^*)$. The implicit function theorem can be applied at all points of Γ, except (λ^*, x^*), however at this point we cannot guarantee uniqueness of the solution. The radii of the balls of uniqueness given by the theorem shrink to zero as (λ^*, x^*) is approached along Γ and there may be a sequence $\{(\mu_i, y_i)\} \notin \Gamma$ with (μ_i, y_i) satisfying (1.1), and $\mu_i \to \lambda^*$, $y_i \to x^*$. A simple illustration of this is the linear example $X = R^n$ and

$$G(\lambda, x) = (A - \lambda I)x, \tag{1.2}$$

where A is a linear operator and I is the identity. The explicitly known curve Γ is $(\lambda, 0)$, which is a solution of (1.2) for all λ. However uniqueness only holds if λ is not an eigenvalue of A i.e. if $G_x(\lambda, 0)$ is non-singular. For if λ^* is an eigenvalue of A and ϕ^* a corresponding eigenvector, then $(\lambda^*, a\phi^*)$ is a solution of (1.2) for all real a.

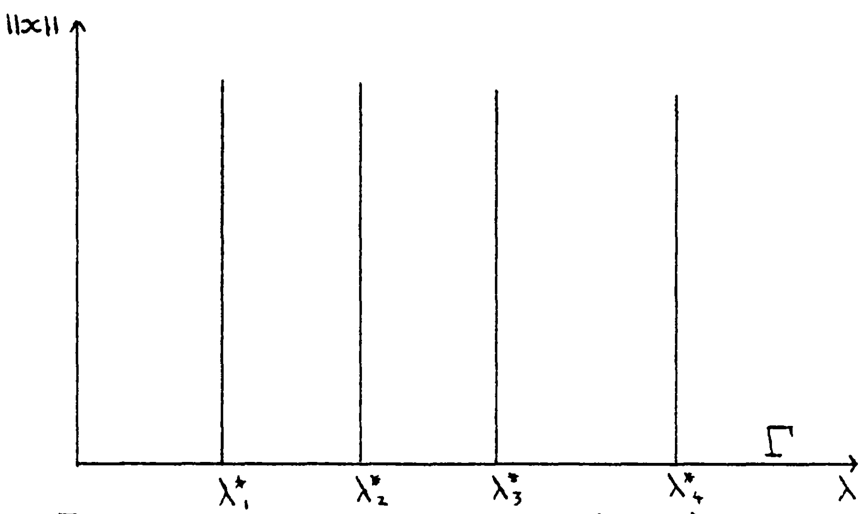

Fig. 1.2 Solution-set of $(A - \lambda I)x = 0$

Returning now to the general non-linear problem (1.1), in many cases either $G(\lambda,0)=0$ for all λ or the equation can easily be changed to this form. The solution curve $(\lambda,0)$ is referred to as the "trivial solution", and if a non-trivial $(x\neq 0)$ solution curve branches away from it then this is called "bifurcation from the trivial solution". In Chapter 2 we consider the problem of starting from the trivial solution and computing these non-trivial bifurcating curves.

With other problems, where $G(\lambda,0)$ is not necessarily zero, a solution curve may not be known explicitly, but we are concerned with calculating one. If (λ^0,x^0) is a solution of (1.1) and $G_x(\lambda^0,x^0)$ is invertible, then the implicit function theorem not only gives the existence and uniqueness of a solution curve $(\lambda,x(\lambda))$, as in Fig. 1.1, but can also be used to construct it. There are many other ways of computing this curve, most being derived by differentiating (1.1) to obtain

$$G_x(\lambda,x(\lambda))\frac{d}{d\lambda}x(\lambda)+G_\lambda(\lambda,x(\lambda))=0\ . \tag{1.3}$$

As $G_x(\lambda,x)$ is invertible near (λ^0,x^0) this equation can be written as an initial-value problem

$$\begin{aligned}\frac{dx(\lambda)}{d\lambda}&=-G_x(\lambda,x(\lambda))^{-1}G_\lambda(\lambda,x(\lambda))\\ x(\lambda^0)&=x^0\ ,\end{aligned}\tag{1.4}$$

and then, under suitable smoothness conditions on G, any of the methods for solving ordinary differential equations may be used. Additionally we have (1.1), and thus a popular predictor-corrector pair is Euler's method applied to (1.4) and Newton's method applied to (1.1).

a) $$\begin{aligned}x^0(\lambda^0+\delta\lambda)&=x(\lambda^0)+\delta\lambda\,\frac{d}{d\lambda}x(\lambda^0)\\ &=x^0-\delta\lambda\,G_x(\lambda^0,x^0)^{-1}G_\lambda(\lambda^0,x^0)\end{aligned}\tag{1.5}$$

b) $G_x(\lambda^0+\delta\lambda, x^\tau(\lambda^0+\delta\lambda))\,\delta x^\tau = -G(\lambda^0+\delta\lambda, x^\tau(\lambda^0+\delta\lambda))$

$$x^{\tau+1}(\lambda^0+\delta\lambda) = x^\tau(\lambda^0+\delta\lambda) + \delta x^\tau \qquad \tau = 0,1,2,\ldots$$

with $x^\tau(\lambda^0+\delta\lambda) \to x(\lambda^0+\delta\lambda)$ as $\tau \to \infty$.

The crucial question with these continuation methods is step-control i.e. the choice of the increment $\delta\lambda$. However if this is adequately answered the method can be used successfully to construct the solution-curve $(\lambda, x(\lambda))$ until a point (λ^*, x^*) is approached at which $G_x(\lambda, x)$ is not invertible. For theoretical convergence $\delta\lambda$ must then be taken smaller and smaller and the singular point is never actually reached. In practice the continuation method is likely to try to "jump over" the singularity and will probably break down or converge only slowly according to the type of singularity encountered.

One type of singularity only occurs because we are trying to parametrise the solution curve using λ and could disappear if some other parametrisation were used. For example suppose the solution curve is expressed as $(\lambda(s), x(s))$, with $(\lambda^*, x^*) = (\lambda(s^*), x(s^*))$, where s is an independent parameter. The analogue of (1.3) is

$$G_x(\lambda(s), x(s))\,\frac{dx}{ds}(s) + G_\lambda(\lambda(s), x(s))\,\frac{d\lambda}{ds}(s) = 0, \qquad (1.6)$$

and if $G_x(\lambda(s), x(s))$ is invertible, (1.6) gives a unique normalised solution for $\left(\frac{d}{ds}\lambda(s), \frac{d}{ds}x(s)\right)$. However if the null-space of $G_x(\lambda^*, x^*)$ is 1 - dimensional and $G_\lambda(\lambda^*, x^*)$ is not in the range of $G_x(\lambda^*, x^*)$ (1.6) still has a unique normalised solution, but with $\frac{d}{ds}\lambda(s^*) = 0$. Under additional assumptions it can be shown that there is a unique solution curve passing through (λ^*, x^*), but it is "perpendicular" to the λ-axis at (λ^*, x^*) $\left(\frac{d\lambda}{ds}(s^*) = 0\right)$. The type of behaviour which may occur is shown in Figure 1.3.

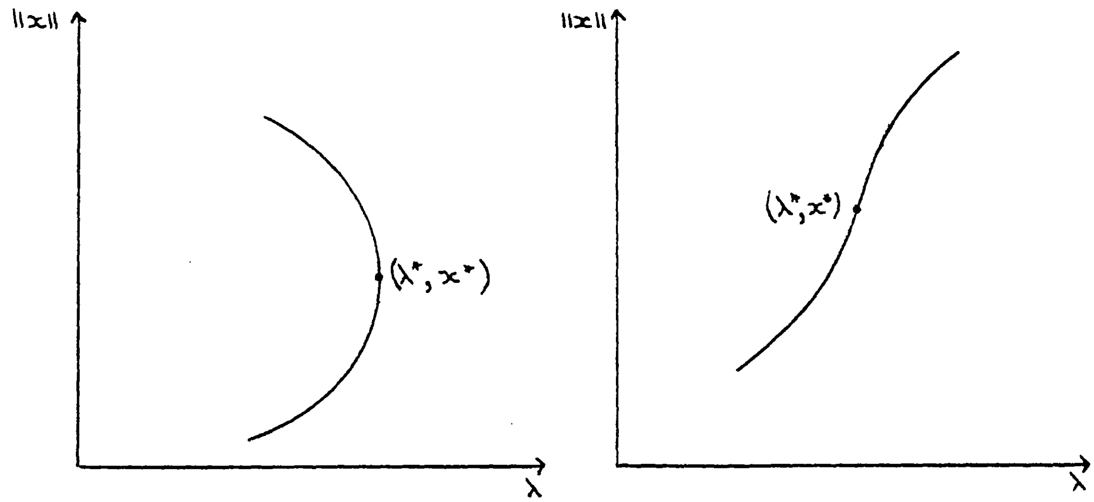

Fig. 1.3 a) Quadratic Type b) Cubic Type

In these cases (λ^*, x^*) is called a TURNING POINT. In particular, in Fig. 1.3 a), (λ^*, x^*) is called a LIMIT POINT as solutions only exist locally for $\lambda \leq \lambda^*$. In Chapter 3 we examine the problem of computing turning points and the curves passing through them.

Another type of behaviour is possible if $G_\lambda(\lambda^*, x^*)$ is in the range of $G_x(\lambda^*, x^*)$. If we again assume that the null-space of $G_x(\lambda^*, x^*)$ is 1 - dimensional, spanned by ϕ^* say, and that $G_x(\lambda^*, x^*)$ is 1-1 from some complement N of $\{\phi^*\}$ in X onto the range of $G_x(\lambda^*, x^*)$, then (1.6) has a two dimensional solution-space

$$\frac{dx}{ds} = \alpha_1 \phi^* + \alpha_2 w^* \qquad \frac{d\lambda}{ds} = \alpha_2 , \tag{1.7}$$

where α_1, α_2 are arbitrary constants and w^* is the unique member of N satisfying

$$G_x(\lambda^*, x^*) w^* = -G_\lambda(\lambda^*, x^*) . \tag{1.8}$$

With additional assumptions on G one can show that there are two distinct solution curves, Γ_1 and Γ_2, passing through (λ^*, x^*).

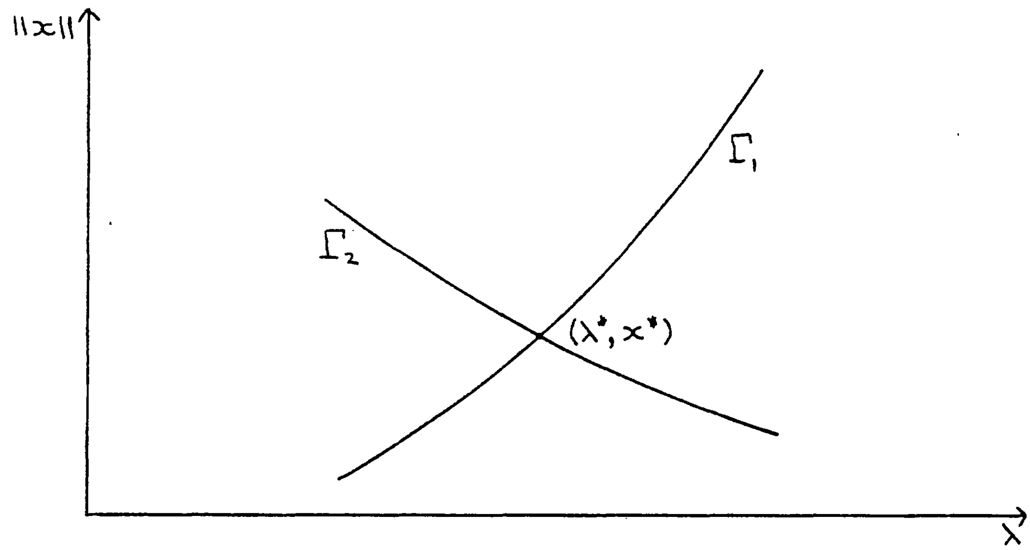

Fig. 1.4 Secondary Bifurcation Point

In this case (λ^*, x^*) is usually called a secondary BIFURCATION POINT, to distinguish between bifurcation points on the trivial solution, which are usually referred to as primary bifurcation points. In Chapter 4 we consider the problem of computing secondary bifurcation points and the different solution curves passing through them.

Of course, in solving equations of the form (1.1) numerically, we must construct finite dimensional approximations $\underset{\sim}{x}$ and $\underset{\sim}{G}$ to x and G and consider the equation

$$\underset{\sim}{G}(\lambda, \underset{\sim}{x}) = 0 , \tag{1.9}$$

where $\underset{\sim}{x} \in R^n$ and $\underset{\sim}{G} : R \times R^n \to R^n$. If secondary bifurcation, as in Figure 1.4, occurs for equation (1.1), this may well be destroyed for the approximation (1.9), and the following types of behaviour can appear.

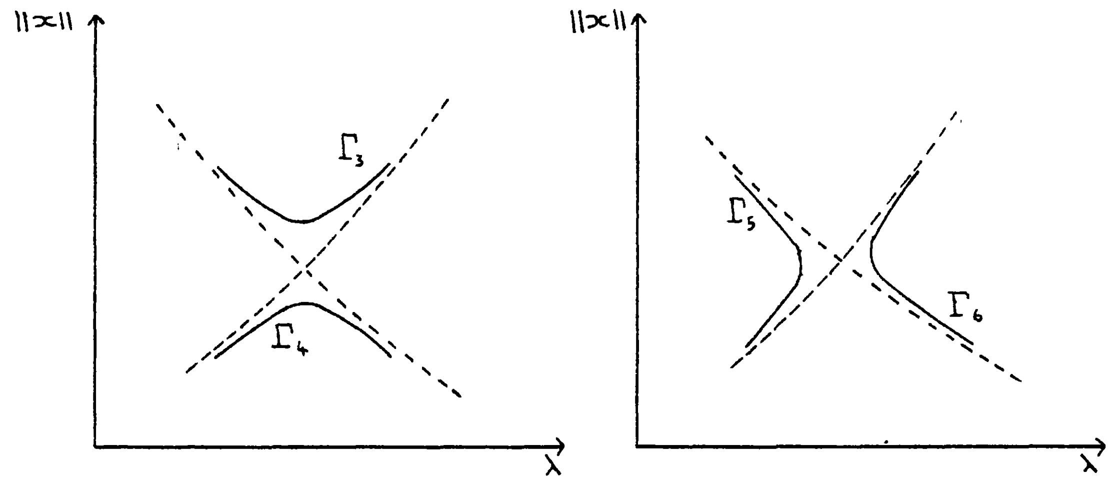

Fig. 1.5 Imperfect Bifurcation

Comparing Figures 1.4 and 1.5 we see that bifurcation no longer occurs as the solution curves do not cross. In fact Γ_3 consists of approximations to a half-branch of Γ_1 and Γ_2 and similarly for Γ_4, Γ_5 and Γ_6. We term this behaviour imperfect bifurcation and describe, in Chapter 5, how the solution curves may be computed and how one may "jump" from one curve to another.

The main technique used to solve these various bifurcation equations is the Newton-Kantorovich method. Of course we are always working near a point (λ^*, x^*) for which $G_x(\lambda^*, x^*)$ is not invertible, and this demands careful consideration. In determining turning points and bifurcation points we modify the equations under consideration so that we are dealing with non-singular operators. In computing the solution curves near singular points we derive accurate starting approximations which compensate for the near-singularity.

CHAPTER 2

BIFURCATION FROM THE TRIVIAL SOLUTION

2.1 Introduction

Throughout this chapter we consider the solution of

$$G(\lambda, x) = 0, \tag{2.1.1}$$

where G is a continuously differentiable operator from R x X to X and $G(\lambda, 0)=0$ for all λ. We are interested in solutions of (2.1.1), with non-zero x-component, near a point $(\lambda^*, 0)$ for which $G_x(\lambda^*, 0)$ is not invertible. In contrast to later chapters we assume that singular points on the trivial solution are relatively easy to compute, and we do not discuss this problem.

If $\mathcal{N}\{L\}$ denotes the null-space, $\mathcal{R}\{L\}$ the range, and L' the conjugate (Taylor [29], p. 213) of a bounded linear operator L, the following assumptions are made about $G_x(\lambda^*, 0)$.

a) $\mathcal{N}\{G_x(\lambda^*, 0)\}$ is 1 - dimensional and spanned by $\phi^*, \|\phi^*\|=1$.

b) $\mathcal{N}\{G_x(\lambda^*, 0)'\}$ is 1 - dimensional and spanned by $\psi^*, \|\psi^*\|=1$.

c) $\mathcal{R}\{G_x(\lambda^*, 0)\}$ is closed. (2.1.2)

Under these assumptions we can show that a restriction of $G_x(\lambda^*, 0)$ is invertible. Let l^* be a bounded linear functional, of unit norm, and $z^* \in X$ such that

a) $l^*(\phi^*) \neq 0$ (2.1.3)

b) $\psi^*(z^*) = 1$.

The existence of such elements follows from the Hahn-Banach theorem. Defining the projections

$$\text{a)} \quad Qx = x - \frac{l^*(x)}{l^*(\phi^*)}\phi^* \tag{2.1.4}$$

b) $Px = x - \psi^*(x) z^*$,

we may decompose X into the direct sums

$$\text{a)} \quad \mathcal{N}\{G_x(\lambda^*,0)\} \oplus N \tag{2.1.5}$$

$$\text{b)} \quad \{z^*\} \oplus \mathcal{R}\{G_x(\lambda^*,0)\} ,$$

where N is the range of Q, or equivalently the subspace of X annihilated by l^*. Now, because $G_x(\lambda^*,0)$ is 1-1 on N, $G_x(\lambda^*,0)$ has a bounded inverse M, by the open-mapping theorem, when considered as an operator from N to $\mathcal{R}\{G_x(\lambda^*,0)\}$.

In certain cases there is a natural choice of l^* and N. If $\psi^*(\phi^*) \neq 0$ so that 0 is a simple eigenvalue of $G_x(\lambda^*,0)$, $l^* = \psi^*$ and $N = \mathcal{R}\{G_x(\lambda^*,0)\}$ is an obvious possibility. If X is a Hilbert space and $G_x(\lambda^*,0)$ self-adjoint then we may take $N = \{\phi^*\}^{\perp}$, where $\perp$ is the orthogonal complement. However to simplify (2.1.4a) we shall in future assume that $l^*(\phi^*) = 1$.

The conditions above, (2.1.2), are not, however, sufficient for bifurcation to occur at $(\lambda^*,0)$. A simple counter-example is $X = R$ and

$$G(\lambda,x) = (\lambda^2+1)x^2 \tag{2.1.6}$$

for which $(\lambda,0)$ is the only solution. Thus we need an extra condition on G.

In example (2.1.6) 0 is an eigenvalue of $G_x(\lambda,0)$ for all λ. If we eliminate this possibility by assuming that for λ close to, but not equal to, λ^*, 0 is not an eigenvalue of $G_x(\lambda,0)$, then topological degree theory can be used to show that bifurcation occurs. However this approach is non-constructive and no uniqueness of the non-trivial solutions near $(\lambda^*,0)$ can be asserted. For example, consider $X = R$ and

$$G(\lambda,x) = x\prod_{i=1}^{n}(x-\lambda^i) , \tag{2.1.7}$$

which has n non-trivial solutions

$$x_i(\lambda) = \lambda^i \qquad i = 1, \dots, n \tag{2.1.8}$$

passing through $(\lambda^*, 0) = (0,0)$.

This digression was meant to show the importance of an additional condition on $G(\lambda, x)$ to ensure the existence of a unique bifurcating solution-curve. The condition that we choose is that the mixed second-order derivative, $G_{\lambda x}(\lambda, x)$ exists, is continuous, and

$$G_{\lambda x}(\lambda^*, 0)\phi^* \notin \mathcal{R}\{G_x(\lambda^*, 0)\}, \tag{2.1.9}$$

or equivalently $\psi^*(G_{x\lambda}(\lambda^*, 0)\phi^*) \neq 0$. As the following theorem shows, this assumption not only guarantees a punctured neighbourhood of λ^* in which $G_x(\lambda, 0)$ is non-singular, but also bounds the rate at which $\|G_x(\lambda, 0)^{-1}\| \to \infty$ as $\lambda \to \lambda^*$. However first we prove a very useful lemma which will be referred to throughout the thesis.

Lemma 2.1

Let L be a bounded linear operator satisfying (2.1.2) and its consequences, and L_1 a bounded linear operator such that $b = |\psi^*(L_1\phi^*)| \neq 0$. If

$$0 < |\alpha| < b\{\|M\|\|P\|\|L_1\|(b + \|L_1\|)\}^{-1} \tag{2.1.10}$$

then $L + \alpha L_1$ has a bounded inverse on X and

$$\|(L + \alpha L_1)^{-1}\| = O(|\alpha|^{-1}). \tag{2.1.11}$$

Proof

Using (2.1.3) and (2.1.4) the equation

$$(L + \alpha L_1)x = y \tag{2.1.12}$$

is equivalent to the pair of equations

$$\begin{aligned} &\text{a)} \quad \alpha l^*(x)\,\psi^*(L_1\phi^*) + \alpha\psi^*(L_1 Qx) = \psi^*(y) \\ &\text{b)} \quad Lx + \alpha P L_1 x = Py\,. \end{aligned} \tag{2.1.13}$$

Solving (2.1.13a) for $l^*(x)$ in terms of y and Qx and inserting into (2.1.13b) gives

$$\begin{aligned}
&\text{a)} \quad l^*(x) = \left(\alpha^{-1}\psi^*(y) - \psi^*(L_1 Qx)\right) / \psi^*(L_1\phi^*) \\
&\text{b)} \quad LQx + \alpha\{PL_1Qx - (\psi^*(L_1Qx)/\psi^*(L_1\phi^*))PL_1\phi^*\} \\
&\qquad = Py - (\psi^*(y)/\psi^*(L_1\phi^*))PL_1\phi^* .
\end{aligned} \tag{2.1.14}$$

Because of the restriction on $|\alpha|$ in the lemma we may use the Banach perturbation theorem (Taylor [29], p. 164) to invert (2.1.14b) and obtain Qx in terms of y. Thus, by construction, $L+\alpha L_1$ is $1-1$ and onto X and therefore, by the open-mapping theorem, it has a bounded inverse. (2.1.11) follows readily from (2.1.14)∎

<u>Theorem 2.2</u>

If (2.1.9) holds then $\exists\ \delta>0$ and $K>0$ such that, for $0<|\lambda-\lambda^*|<\delta$, $G_x(\lambda,0)^{-1}$ exists and

$$\|G_x(\lambda,0)^{-1}\| < K|\lambda-\lambda^*|^{-1}. \tag{2.1.15}$$

<u>Proof</u>

We may expand $G_x(\lambda,0)$ about $(\lambda^*,0)$ to obtain

$$G_x(\lambda,0) = G_x(\lambda^*,0) + (\lambda-\lambda^*)G_{x\lambda}(\lambda^*,0) + R(\lambda-\lambda^*) \tag{2.1.16}$$

where $\|R(\lambda-\lambda^*)\| = o(|\lambda-\lambda^*|)$. The theorem is then proved by applying the previous lemma to the operator $G_x(\lambda^*,0)+(\lambda-\lambda^*)G_{x\lambda}(\lambda^*,0)$ and using the Banach perturbation theorem to cater for the higher order term $R(\lambda-\lambda^*)$∎

The importance of (2.1.9) is shown in the next section where we prove that there is a unique non-trivial solution curve of (2.1.1) bifurcating away from $(\lambda^*,0)$.

To conclude this section we note that, in the early investigations into bifurcation theory, interest was focused on integral equations of the form

$$G(\lambda,x) = (\lambda I - K)x \tag{2.1.17}$$

where K is a completely continuous operator. If $\lambda^*\neq 0$ is a simple eigenvalue of $K_x(0)$ then (2.1.2b) and (2.1.2c) follow from the

Fredholm theory of such operators, and (2.1.9) holds since $G_{\lambda x}(\lambda,0)=I$. Thus such results are special cases of ours.

In section 2.2 we introduce the Liapunov-Schmidt method, which is used throughout the thesis to obtain solutions of (1.1) near singular points. More recent constructive methods are briefly described in section 2.3, and in section 2.4 we develop Newton's method in the form we require. Finally, in sections 2.5, 2.6 and 2.7, Newton's method is applied in various ways to obtain the bifurcating solutions. Numerical results are given in 2.8.

2.2 The Liapunov-Schmidt Reduction

In this section we describe a classical method for proving the existence and uniqueness of non-trivial bifurcating solutions of (2.1.1). It originated in the work of Liapunov [27], and has since been greatly generalised into what is known as the alternative method (Cesari [5]).

From our assumptions (2.1.2) and (2.1.9) in the previous section we may expand $G(\lambda,x)$ about $(\lambda^*,0)$ to obtain

$$G(\lambda,x)=G_x(\lambda^*,0)x+(\lambda-\lambda^*)G_{x\lambda}(\lambda^*,0)x+R_1(\lambda,x), \qquad (2.2.1)$$

where $R_1(\lambda,0)=0$ for all λ, and R_1 has continuous first-order derivatives and a continuous mixed second-order derivative, all of which are zero at $(\lambda^*,0)$. Now we decompose X, using (2.1.5a), and seek solutions of

$$G(\lambda^*+\delta\lambda,\varepsilon\phi^*+w)=0, \qquad (2.2.2)$$

where $w\in N$ and $\delta\lambda$, ε and w are "small". Expressing (2.2.2) in the form of (2.2.1) gives

$$G_x(\lambda^*,0)w=-\delta\lambda G_{x\lambda}(\lambda^*,0)(\varepsilon\phi^*+w)-R_1(\lambda^*+\delta\lambda,\varepsilon\phi^*+w), \qquad (2.2.3)$$

and for this equation to be solvable the right-hand side must be in the range of $G_x(\lambda^*,0)$. If this is so then the "inverse" M of $G_x(\lambda^*,0)$, developed in the previous section, may be used to determine w. Thus, using the projection P of (2.1.4b), (2.2.3) is equivalent to the pair of equations

$$\begin{aligned} &\text{a)}\quad w+\delta\lambda MPG_{\lambda x}(\lambda^*,0)(\varepsilon\phi^*+w)+MPR_1(\lambda^*+\delta\lambda,\varepsilon\phi^*+w)=0 \\ &\text{b)}\quad \delta\lambda\psi^*(G_{\lambda x}(\lambda^*,0)(\varepsilon\phi^*+w))+\psi^*(R_1(\lambda^*+\delta\lambda,\varepsilon\phi^*+w))=0\,. \end{aligned} \tag{2.2.4}$$

(2.2.4a) can be written in the form

$$w+T(\delta\lambda,\varepsilon,w)=0 \tag{2.2.5}$$

with $T : R \times R \times N \to N$ a continuously differentiable operator and

$$\frac{\partial T}{\partial w}(0,0,0)=0\;, \tag{2.2.6}$$

and thus the implicit function theorem can be applied to show that (2.2.4a) has a unique small solution $w(\delta\lambda,\varepsilon)$ for sufficiently small $\delta\lambda$ and ε. By definition of T, $w(\delta\lambda,\varepsilon)$ also has continuous first-order derivatives and a continuous mixed second-order derivative with

$$\frac{\partial w}{\partial\delta\lambda}(0,0)=\frac{\partial w}{\partial\varepsilon}(0,0)=0 \tag{2.2.7}$$

and hence

$$\|w\|=o(|\delta\lambda|+|\varepsilon|)\,. \tag{2.2.8}$$

Having solved (2.2.4a) we may insert the solution $w(\delta\lambda,\varepsilon)$ into (2.2.4b) to obtain

$$a\,\delta\lambda\,\varepsilon+R_2(\delta\lambda,\varepsilon)=0\,, \tag{2.2.9}$$

where $a=\psi^*(G_{\lambda x}(\lambda^*,0)\phi^*)$ is non-zero by (2.1.9). R_2 is a higher-order operator from $R^2 \to R$ with $R_2(\delta\lambda,0)=0$ for all $\delta\lambda$ and

$$\frac{\partial R_2}{\partial\delta\lambda}(0,0)=\frac{\partial R_2}{\partial\varepsilon}(0,0)=\frac{\partial^2 R_2}{\partial\delta\lambda\,\varepsilon}(0,0)=0\,. \tag{2.2.10}$$

Of course (2.2.9) has the solution $\varepsilon=0$, $\delta\lambda$ arbitrary, and through (2.2.4a) this gives $w(\delta\lambda,\varepsilon)=0$ and so corresponds to the trivial solution. We divide this out to give

$$a\,\delta\lambda+R_3(\delta\lambda,\varepsilon)=0 \tag{2.2.11}$$

where $R_3(\delta\lambda,\varepsilon)=\varepsilon^{-1}R_2(\delta\lambda,\varepsilon)$. Using (2.2.10) we see that R_3 is continuous with respect to ε and continuously differentiable with respect to $\delta\lambda$, with

$$R_3(0,0)=\frac{\partial R_3}{\partial\delta\lambda}(0,0)=0\,. \tag{2.2.12}$$

Thus the implicit function theorem can be applied to (2.2.11) to give a unique continuous solution $\delta\lambda(\varepsilon)$ for $|\varepsilon|$ sufficiently small. Inserting this into the solution $w(\delta\lambda,\varepsilon)$ of (2.2.4a) shows that (2.2.2) has a unique non-trivial solution curve

$$\begin{aligned} &a) \quad \lambda^*(\varepsilon) = \lambda^* + \delta\lambda(\varepsilon) \\ &b) \quad x^*(\varepsilon) = \varepsilon\phi^* + w(\delta\lambda(\varepsilon),\varepsilon) \end{aligned} \tag{2.2.13}$$

bifurcating from $(\lambda^*,0)$.

As the implicit function theorem is itself constructive the above method can be used to actually compute the solution-curve. The calculation consists of two iterations, an outer loop to solve (2.2.4b), and an inner loop to solve (2.2.4a).

$$\begin{aligned} &\delta\lambda^0 = 0 \qquad w^0 = 0 \\ &w^{n+1} = \lim v^k \qquad \text{where} \\ &\qquad v^0 = w^n \\ &\qquad v^{k+1} = \delta\lambda^n MPG_{\lambda x}(\lambda^*,0)(\varepsilon\phi^* + v^k) \\ &\qquad\qquad + MPR_1(\lambda^*+\delta\lambda^n, \varepsilon\phi^*+v^k) \\ &\delta\lambda^{n+1} = \delta\lambda^n - (a\varepsilon)^{-1}\{\delta\lambda^n\psi^*(G_{\lambda x}(\lambda^*,0)w^{n+1}) \\ &\qquad\qquad + \psi^*(R_1(\lambda^*+\delta\lambda^n, \varepsilon\phi^*+w^{n+1}))\}. \end{aligned} \tag{2.2.14}$$

Replacing $R_1(\delta\lambda,x)$ by $G(\lambda,x) - G_x(\lambda^*,0)x - (\lambda-\lambda^*)G_{\lambda x}(\lambda^*,0)x$ gives the simpler form

$$\begin{aligned} &\delta\lambda^0 = 0 \qquad w^0 = 0 \\ &w^{n+1} = \lim v^k \qquad \text{where} \\ &\qquad v^0 = w^n \\ &\qquad v^{k+1} = v^k - MPG(\lambda^*+\delta\lambda^n, \varepsilon\phi^*+v^k) \\ &\delta\lambda^{n+1} = \delta\lambda^n - (a\varepsilon)^{-1}\psi^*(G(\lambda^*+\delta\lambda^n, \varepsilon\phi^*+w^{n+1})). \end{aligned} \tag{2.2.15}$$

2.3 Constructive Methods for Bifurcating Solutions

The Liapunov-Schmidt method described in the previous section is not very efficient for practical purposes as it requires the solution of a non-linear system of equations at each iteration. Here we briefly describe two methods which have been developed more recently and which do not have this drawback.

2.3.1 Keller's Iterative Method

This procedure was originally developed for the solution of differential equations in Keller [16] and has since been generalised by Keller and Langford [19].

The basic idea is to reverse the order of the Liapunov-Schmidt method and first solve (2.2.4b) for λ. This value can then be inserted into (2.2.4a) and this equation solved for w. This means that only one non-linear scalar equation need be solved at each iteration. Using the notation of the previous section the iteration is

$$\begin{aligned} &\delta\lambda^0 = 0 \qquad w^0 = 0 \\ &\delta\lambda^{n+1} = \lim \mu^k \\ &\qquad \mu^0 = \delta\lambda^n \\ &\qquad \mu^{k+1} = \mu^k - (a\varepsilon)^{-1}\psi^*\left(G(\lambda^*+\mu^k, \varepsilon\phi^*+w^n)\right) \\ &w^{n+1} = w^n - MG(\lambda^*+\delta\lambda^{n+1}, \varepsilon\phi^*+w^n). \end{aligned} \tag{2.3.1}$$

The projection P is not needed as the inner iteration ensures that the right-most component of the last equation is in the domain of M.

In Keller and Langford the iteration is changed slightly to improve convergence and additional assumptions are made about the higher-order operator R to give convergence rates for the iterates in terms of ε.

The above iteration still has the disadvantage that a non-linear scalar equation must be solved at each step, but a modification of Demoulin and Chen [6] avoids this. They re-write (2.2.2) in the form

$$G_x(\lambda^*,0)w = G(\lambda^*+\delta\lambda, \varepsilon\phi^*+w) + G_x(\lambda^*,0)w + \varepsilon(\lambda-\lambda)G_{x\lambda}(\lambda^*,0)\phi^* , \tag{2.3.2}$$

and this gives the iteration

$$\begin{aligned} &\delta\lambda^0 = 0 \qquad w^0 = 0 \\ &\delta\lambda^{n+1} = \delta\lambda^n - (\varepsilon a)^{-1}\psi^*(G(\lambda^*+\delta\lambda^n, \varepsilon\phi^*+w)) \\ &w^{n+1} = w^n - M\{G(\lambda^*+\delta\lambda^n, \varepsilon\phi^*+w^n) + \varepsilon(\delta\lambda^{n+1}-\delta\lambda^n)G_{\lambda x}(\lambda^*,0)\phi^*\} \end{aligned} \tag{2.3.3}$$

which just consists of a linear system at each step.

2.3.2 Implicit Function Method

In [8] Crandall and Rabinowitz modify (2.2.2) and then use the implicit function theorem to prove very general results about the occurrence of bifurcation. This method can also be used to compute the bifurcating solutions. As the implicit function theorem guarantees the uniqueness of solutions the trivial solution must be removed and this is achieved by solving (2.2.2) for λ and w in terms of ε and "dividing out" the trivial solution $\varepsilon = 0$. Thus, writing $y = (\lambda, w)$ we define

$$H(y,\varepsilon) = \varepsilon^{-1}G(\lambda, \varepsilon(\phi^*+w)) \tag{2.3.4}$$

for $\varepsilon \neq 0$, where $H : (R \times N) \times R \to X$. This definition can be extended to $\varepsilon = 0$ by continuity to give

$$H(y,0) = G_x(\lambda,0)(\phi^*+w) \tag{2.3.5}$$

Now, with $y^* \doteq (\lambda^*,0)$, $H(y^*,0) = 0$ and we wish to use the implicit function theorem to construct a non-trivial solution $y(\varepsilon)$ with $y(0) = y^*$. Letting $z = (\mu, v) \in R \times N$ the derivative of H is defined by

$$\begin{aligned} &\text{a)} \quad H_y(y,\varepsilon)z = \mu\varepsilon^{-1}G_\lambda(\lambda, \varepsilon(\phi^*+w)) + G_x(\lambda, \varepsilon(\phi^*+w))v \\ &\qquad\qquad \text{for } \varepsilon \neq 0 \\ &\text{b)} \quad H_y(y,0)z = \mu G_{\lambda x}(\lambda,0)(\phi^*+w) + G_x(\lambda,0)v , \end{aligned} \tag{2.3.6}$$

and so H is continuously differentiable with respect to y and $H_y(y^*,0)$ is invertible by (2.1.9). Thus the implicit function theorem can be used to construct the bifurcating solution $(y(\varepsilon),\varepsilon)$ by means of the iteration

$$y^{n+1}(\varepsilon) = y^n(\varepsilon) - H_y(y^*,0)^{-1} H(y^n(\varepsilon),\varepsilon) \qquad (2.3.7)$$

with $y^0(\varepsilon) = y^*$.

Expressing this in terms of the original operator G gives

$$\begin{aligned} &\lambda^0 = \lambda^* \qquad w^0 = 0 \qquad\qquad (2.3.8)\\ &\lambda^{n+1} = \lambda^n - (\varepsilon a)^{-1} \psi^*(G(\lambda^n, \varepsilon(\phi^* + w^n)))\\ &w^{n+1} = w^n - M\{\varepsilon^{-1} G(\lambda^n, \varepsilon(\phi^*+w^n)) + (\lambda^{n+1}-\lambda^n) G_{x\lambda}(\lambda^*,0)\phi^*\}. \end{aligned}$$

It can be seen that this is the same iteration as (2.3.3), Demoulin and Chen's modification to Keller's method.

2.4 Newton's Method

In this section we state the standard theorems on the convergence of the Newton- Kantorovich method and its modified form, and then develop corollaries which can be applied more easily to bifurcation problems.

Throughout this section Z_1 and Z_2 denote general Banach spaces and the following theorem, due to Kantorovich (Kantorovich and Akilov [14], Krasno'selskii [38]), is basic for the convergence of Newton's method.

Theorem 2.3

Let $S(x)$ be a differentiable mapping from Z_1 to Z_2 and x^0 an element of Z_1 such that $S_x(x^0)$ has a bounded inverse, which satisfies

(i) $\|S_x(x^0)^{-1} S(x^0)\| \leq \eta$

(ii) $\|S_x(x^0)^{-1}\{S_x(x) - S_x(y)\}\| \leq L\|x-y\|$

for all x, y s.t. $\|x-x^0\| < R$, $\|y-y^0\| < R$.

Then, if $h = L\eta \leq \frac{1}{2}$ and $\eta(1-(1-2h)^{1/2})/h \leq R$, the equation $S(x)=0$ has a solution x^* to which the Newton sequence

$$x^{n+1} = x^n - S_x(x^n)^{-1} S(x^n) \qquad n = 0,1,2,\ldots$$

converges, and the rate of convergence is given by

$$\|x^n - x^*\| \leq (2h)^{2^n-1} \eta/2^{n-1} .$$

If $R < \eta(1+(1-2h)^{1/2})/h$ then the solution x^* is unique in the ball $B(x^0,R)$.

We note that, for $h<\frac{1}{2}$, $R=2\eta$ satisfies the inequalities for R in the theorem.

For bifurcation problems we will have a non-linear function depending on a real parameter, δ say, and our equation will take the form

$$T(\delta,x) = 0 \qquad T: R\times Z_1 \to Z_2 . \tag{2.4.1}$$

If we denote the initial approximation for Newton's method by $x^0(\delta)$ then, for the examples we shall consider, $x^0(0)=0$ and $T_x(0,0)^{-1}$ will not exist. However $T_x(\delta,x^0(\delta))^{-1}$ will exist for δ in a sufficiently small punctured neighbourhood of zero with

$$\|T_x(\delta,x^0(\delta))^{-1}\| \to \infty \quad \text{as} \quad |\delta| \to 0 . \tag{2.4.2}$$

Nevertheless a suitable choice of $x^0(\delta)$ may well lead to the corresponding $\eta(\delta)$ and $L(\delta)$ of Theorem 2.3 satisfying $|\eta(\delta)L(\delta)| \to 0$ as $|\delta| \to 0$ and the theorem holding for sufficiently small non-zero δ. These ideas are embodied in the following corollary where we restrict δ to be positive, the negative case follows similarly.

<u>Corollary 2.4</u>

Let $T(\delta,x)$ be a mapping from $R^+\times Z_1$ to Z_2 which is continuous in the first variable and continuously differentiable in the second. Let $x^0(\delta)$ be a continuous mapping from R^+ to Z_1 such that

(i) $T_x(\delta,x^0(\delta))^{-1}$ exists for $0<\delta<\delta_1$

(ii) $\|T_x(\delta, x^0(\delta))^{-1} T(\delta, x^0(\delta))\| \leq \eta(\delta)$ for $0 < \delta < \delta_1$

(iii) $\|T_x(\delta, x^0(\delta))^{-1}\{T_x(\delta, y^1) - T_x(\delta, y^2)\}\| \leq L(\delta)\|y^1 - y^2\|$

for $\|y^i - x^0(\delta)\| < 2\eta(\delta)$ $i = 1,2$ and $0 < \delta < \delta_1$

(iv) $\eta(\delta)L(\delta) \to 0$ as $\delta \to 0$.

Then there exists $\delta_2 > 0, \delta_2 < \delta_1$ such that for $0 < \delta < \delta_2$

a) $T(\delta, x) = 0$ has a continuous solution $x^*(\delta)$

b) $x^*(\delta)$ is unique in the ball $B(x^0(\delta), 2\eta(\delta))$

c) the Newton iteration

$$x^{n+1}(\delta) = x^n(\delta) - T_x(\delta, x^n(\delta))^{-1} T(\delta, x^n(\delta)) \qquad n = 0,1,2,\ldots$$

converges to $x^*(\delta)$.

Proof

The corollary follows immediately from Theorem 2.3 because $h(\delta) < \frac{1}{2}$ for sufficiently small δ ∎

The convergence rate of Theorem 2.3 also holds and we note that, if the speed with which $h(\delta) \to 0$ is known e.g. $h(\delta) = O(\delta^m)$ then it can be expressed in terms of δ.

We now move on to consider the modified Newton method, whose convergence is given by the following standard theorem.

Theorem 2.5

Let $S(x)$ be a differentiable mapping from Z_1 to Z_2 and x^0 an element of Z_1, for which $S_x(x^0)^{-1}$ exists, which satisfies

(i) $\|S_x(x^0)^{-1} S(x^0)\| \leq \eta$

(ii) $\|S_x(x^0)^{-1}\{S(x) - S(y)\}\| \leq L\|x - y\|$

for all x, y such that $\|x - x^0\| < R$ and $\|y - x^0\| < R$.

Then, if $h = \eta L < \frac{1}{2}$ and $\eta(1 - (1-2h)^{1/2})/h \leq R$, the equation $S(x) = 0$ has a solution x^* to which the modified Newton sequence

$$x^{n+1} = x^n - S_x(x^0)^{-1}S(x^n) \qquad n=0,1,2,\ldots$$

converges, and the rate of convergence is given by

$$\|x^{n+1}-x^*\| \leqslant \eta\,(1-(1-2h)^{1/2})^n/(1-2h)^{1/2} .$$

If $R<\eta(1+(1-2h)^{1/2})/h$ then x^* is unique in the ball $B(x^0,R)$.

In fact the modified Newton method consists simply of considering the operator

$$A(x) = x - S_x(x^0)^{-1}S(x) \tag{2.4.3}$$

as a contraction mapping in the ball $B(x^0,\eta(1-(1-2h)^{1/2})/h)$, with Lipschitz constant $1-(1-2h)^{1/2}$. Thus, when considering its application to (2.4.1), we merely make sufficient assumptions for this to be so.

Theorem 2.6

Let $T(\delta,x)$ be a mapping from $R^+\times Z_1$ to Z_2 which is continuous in the first variable and continuously differentiable in the second. Let $x^0(\delta)$ be a continuous mapping from R^+ to Z_1 such that

(i) $T_x(\delta,x^0(\delta))^{-1}$ exists for $0<\delta<\delta_1$

(ii) $\|T_x(\delta,x^0(\delta))^{-1}T(\delta,x^0(\delta))\| \leqslant \eta(\delta)$ for $0<\delta<\delta_1$

(iii) $\|T_x(\delta,x^0(\delta))^{-1}\{T_x(\delta,x^0(\delta))(y_1-y_2)-(T_x(\delta,y_1)-T(\delta,y_2))\}\|$
$\leqslant q(\delta)\|y_1-y_2\|$

for all y_1, y_2 such that $\|y_i-x^0(\delta)\|<2\eta(\delta)$ $i=1,2$ and for $0<\delta<\delta_1$

(iv) $q(\delta)\rightarrow 0$ as $\delta\rightarrow 0$.

Then there exists $\delta_2>0$, $\delta_2<\delta_1$ such that for $0<\delta<\delta_2$

a) $T(\delta,x)=0$ has a continuous solution $x^*(\delta)$

b) $x^*(\delta)$ is unique in the ball

c) the modified Newton iteration

$$x^{n+1}(\delta) = x^n(\delta) - T_x(\delta,x^0(\delta))^{-1}T(\delta,x^n(\delta)) \qquad n=0,1,2,\ldots$$

converges to $x^*(\delta)$

d) the rate of convergence is given by

$$\|x^n(\delta)-x^*(\delta)\| \leq q(\delta)^n \eta(\delta)/(1-q(\delta)) .$$

Proof

We simply apply the contraction mapping theorem to

$$A(\delta,x) = x - T_x(\delta,x^\circ(\delta))^{-1}T(\delta,x) \tag{2.4.4}$$

with S sufficiently small for $q(\delta) < \frac{1}{2}$ ∎

If the rate at which $q(\delta)\to 0$ is known, and also an expression for $\eta(\delta)$ in terms of δ, then the convergence rate in d) above can be given in terms of δ.

It often occurs that we do not wish to apply the modified Newton method exactly, but merely some approximation to it e.g.

$$x^{n+1}(\delta) = x^n(\delta) - \bar{\bar{T}}(\delta)^{-1}T(\delta,x^n(\delta)) , \tag{2.4.5}$$

where $\bar{\bar{T}}(\delta)$ is a linear operator which approximates $T_x(\delta,x^\circ(\delta))$ in some sense. Because we are just using the contraction mapping theorem, if conditions (i) to (iv) of Theorem 2.6 hold with $T_x(\delta,x^\circ(\delta))$ replaced by $\bar{\bar{T}}(\delta)$ then so do the conclusions.

In section 2.3 we briefly described some of the special methods used to compute non-trivial bifurcating solutions of (2.1.1). Now we wish to apply the Newton-Kantorovich method to this problem, and there are two main ways of implementing it.

Since (2.1.1) is an equation from R x X to X, it is perhaps most natural to fix λ, $\lambda=\lambda^*+\delta\lambda$ say, and then to compute the Newton iterates $\{x^n\}$ by solving equations in X i.e.

$$G_x(\lambda,x^n)(x^{n+1}-x^n) = -G(\lambda,x^n). \tag{2.4.6}$$

Hopefully this will produce the bifurcating solution at $\lambda=\lambda^*+\delta\lambda$ but, as in the linear case (Fig. 1.2), the non-trivial solutions may only exist for $\lambda=\lambda^*$. However, under additional assumptions which prevent this, we show, in sections 2.6 and 2.7, how suitable starting values may be constructed which guarantee convergence. Rates of convergence in terms of $\delta\lambda$ are also developed.

Newton's method can also be applied in the following way. Let X_1 be a one-dimensional subspace of X spanned by x_1, with $\|x_1\|=1$, and X_2 a complement of X_1 in X. Then (2.1.1) can be written in the form

$$G(\lambda,\alpha(x_1+x_2))=0 \tag{2.4.7}$$

with $\alpha\in R$ and $x_2\in X_2$. If α is kept fixed, then (2.4.7) is a mapping from $R\times X_2$ to X and the Newton iterates (λ^n, x_2^n) are given by

$$\begin{aligned} &G_x(\lambda^n,\alpha(x_1+x_2^n))(\alpha x_2^{n+1}-\alpha x_2^n)+\\ &(\lambda^{n+1}-\lambda^n)G_\lambda(\lambda^n,\alpha(x_1+x_2^n))=-G(\lambda^n,\alpha(x_1+x_2^n))\end{aligned} \tag{2.4.8}$$

or, dividing through by α,

$$\begin{aligned} &G_x(\lambda^n,\alpha(x_1+x_2^n))(x_2^{n+1}-x_2^n)+\\ &\alpha^{-1}(\lambda^{n+1}-\lambda^n)G_\lambda(\lambda^n,\alpha(x_1+x_2^n))=-\alpha^{-1}G(\lambda^n,\alpha(x_1+x_2^n)).\end{aligned} \tag{2.4.9}$$

Under certain conditions this will converge to the non-trivial solution whose component in X is α. A natural choice for X_1 is $\mathcal{N}\{G_x(\lambda^*,0)\}$ as we know, from section 2.2, that, for $|\alpha|$ sufficiently small, a non-trivial solution exists whose component in $\mathcal{N}\{G_x(\lambda^*,0)\}$ is α . This method is investigated in section 2.5 and leads to results which link up with those of section 2.3.

In Chapter 4 we look at general bifurcation points, i.e. not necessarily on the trivial solution, and mention how the methods developed there can be specialised to the problems of the present chapter.

2.5 Newton's Method with ϵ as Parameter

In this section iterations of the form (2.4.8), with $X_1 = \mathcal{N}\{G_x(\lambda^*,0)\}$ and $X_2 = N$ are considered. As we are iterating with $(\lambda, w), w \in N$, we introduce the space $Y = R \times N$, which becomes a Banach space under the norm

$$\|y\|_Y = \max\{|\lambda|, \|w\|\} \qquad y = (\lambda, w), \tag{2.5.1}$$

and define the operator $H : R \times Y \to X$ by

$$H(\epsilon, y) = G(\lambda, \epsilon(\phi^* + w)). \tag{2.5.2}$$

We can compute the Newton iterates $y^n(\epsilon) = (\lambda^n(\epsilon), w^n(\epsilon))$ for (2.5.2), which are now shown to converge to the bifurcating solution of (2.1.1), cf. (2.2.13), $(\epsilon, y^*(\epsilon))$.

First we consider the modified Newton method

$$y^{n+1}(\epsilon) = y^n(\epsilon) - H_y(\epsilon, y^0)^{-1} H(\epsilon, y^n(\epsilon)). \tag{2.5.3}$$

Theorem 2.7

For fixed ϵ, $|\epsilon|$ sufficiently small, the Newton iterates (2.5.3), with starting approximation $y^0 = (\lambda^*, 0)$, converge to the bifurcating solution $y^*(\epsilon)$.

Proof

We verify the conditions of Theorem 2.6. Firstly,

$$\begin{aligned} H_y(\epsilon, y^0) &= \epsilon G_x(\lambda^*, \epsilon\phi^*) + G_\lambda(\lambda^*, \epsilon\phi^*) \\ &= \epsilon\{G_x(\lambda^*,0) + G_{\lambda x}(\lambda^*,0)\phi^* \\ &\quad + (G_x(\lambda^*,\epsilon\phi^*) - G_x(\lambda^*,0)) \\ &\quad + (\epsilon^{-1} G_\lambda(\lambda^*,\epsilon\phi^*) - G_{\lambda x}(\lambda^*,0)\phi^*)\} \end{aligned} \tag{2.5.4}$$

and thus, as $G_x(\lambda^*,0) + G_{\lambda x}(\lambda^*,0)\phi^*$ is an invertible operator from Y to X by (2.1.9), and the other terms are of higher order in ε, we may use the Banach perturbation theorem to show that $H_y(\varepsilon, y^\circ)^{-1}$ exists for sufficiently small $|\varepsilon| \neq 0$, and

$$\|H_y(\varepsilon, y^\circ)^{-1}\| = O(|\varepsilon|^{-1}). \tag{2.5.5}$$

Secondly,

$$\begin{aligned} H(\varepsilon, y^\circ) &= G(\lambda^*, \varepsilon\phi^*) \\ &= \varepsilon\left(\varepsilon^{-1} G(\lambda^*, \varepsilon\phi^*) - G_x(\lambda^*, 0)\phi^*\right) \end{aligned} \tag{2.5.6}$$

and so

$$\|H(\varepsilon, y^\circ)\| = o(|\varepsilon|) \tag{2.5.7}$$

and

$$\|H_y(\varepsilon, y^\circ)^{-1} H(\varepsilon, y^\circ)\| \leq K_1(\varepsilon) \tag{2.5.8}$$

where $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thirdly, if $z^i = (\mu^i, v^i)$ $i=1,2$ and $\|z^i - y^\circ\| < 2K_1(\varepsilon)$, then

$$\begin{aligned} &H_y(\varepsilon, y^\circ)(z^1 - z^2) - \left(H(\varepsilon, z^1) - H(\varepsilon, z^2)\right) \\ &= \int_0^1 \left\{H_y(\varepsilon, y^\circ) - H_y(\varepsilon, z^2 + t(z^1 - z^2))\right\}(z^1 - z^2)\,dt \end{aligned} \tag{2.5.9}$$

and, if we write $z_t = z^2 + t(z^1 - z^2)$, and similarly for μ_t and v_t,

$$\begin{aligned} H_y(\varepsilon, y^\circ) - H_y(\varepsilon, z_t) = \varepsilon\{ &G_x(\lambda^*, \varepsilon\phi^*) - G_x(\mu_t, \varepsilon(\phi^* + v_t)) \\ &+ \varepsilon^{-1} G_\lambda(\lambda^*, \varepsilon\phi^*) - G_{\lambda x}(\lambda^*, 0)\phi^* \\ &+ G_{\lambda x}(\lambda^*, 0)\phi^* - G_{\lambda x}(\mu_t, 0)(\phi^* + v_t) \\ &+ G_{\lambda x}(\mu_t, 0)(\phi^* + v_t) - \varepsilon^{-1} G_\lambda(\mu_t, \varepsilon(\phi^* + v_t))\} \end{aligned} \tag{2.5.10}$$

and thus

$$\begin{aligned} &\|H_y(\varepsilon, y^\circ)(z^1 - z^2) - \left(H(\varepsilon, z^1) - H(\varepsilon, z^2)\right)\| \\ &\qquad \leq K_2(\varepsilon)\,|\varepsilon|\,\|z^1 - z^2\| \end{aligned} \tag{2.5.11}$$

where $K_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Combining (2.5.5) and (2.5.11) we see that the conditions of Theorem 2.6 hold and the theorem is proved ■

If more information is known about the operator G we can express the rate of convergence in terms of ε. Suppose

$$G(\lambda, x) = G_x(\lambda, 0)x + R(\lambda, x), \tag{2.5.12}$$

where

$$\|R(\lambda,x)-R(\lambda,z)\| \leq K_3(\lambda)\{\|x\|^{p-1}+\|z\|^{p-1}\}\|x-z\| \tag{2.5.13}$$

with $K_3(\lambda)$ bounded and $p \geq 2$, and

$$\|G_{\lambda x}(\lambda,0)-G_{\lambda x}(\mu,0)\| = O(|\lambda-\mu|); \tag{2.5.14}$$

then the following corollary holds.

Corollary 2.8

Under the additional assumptions (2.5.12-14) the rate of convergence of the modified Newton method in Theorem 2.7 is given by

$$\|y^n(\varepsilon)-y^*(\varepsilon)\| \leq (K_4|\varepsilon|^{p-1})^n K_5|\varepsilon|^{p-1}. \tag{2.5.15}$$

Proof

Using the extra information we can express $K_1(\varepsilon)$ and $K_2(\varepsilon)$ of the previous theorem in terms of ε,

$$K_1(\varepsilon) = O(|\varepsilon|^{p-1}) \quad \text{and} \quad K_2(\varepsilon) = O(|\varepsilon|^{p-1}), \tag{2.5.16}$$

and the corollary follows ■

Expressing (2.5.15) in terms of λ and $x=\varepsilon(\phi^*+w)$, instead of $y=(\lambda,w)$, gives

$$\begin{aligned} &\text{a)} \quad \|x^n(\varepsilon)-x^*(\varepsilon)\| \leq (K_4|\varepsilon|^{p-1})^n K_5|\varepsilon|^{p} \\ &\text{b)} \quad |\lambda^n(\varepsilon)-\lambda^*(\varepsilon)| \leq (K_4|\varepsilon|^{p-1})^n K_5|\varepsilon|^{p-1} \end{aligned} \tag{2.5.17}$$

which are the same rates as Keller's method in section 2.3. If, instead of the exact modified Newton method, we use the approximation to $H_y(\varepsilon,y^0)$ given by

$$\varepsilon G_x(\lambda^*,0) + G_{\lambda x}(\lambda^*,0)\phi^*, \tag{2.5.18}$$

a similar result to Corollary 2.8 follows, with the convergence rate having the same powers of ε. In fact this iterative scheme is just Demoulin and Chen's modification to Keller's method (see (2.3.3)). In this case we do not even require the differentiability of R with respect to x but merely a Lipschitz condition of the form (2.5.13).

For the sake of completeness we briefly give the results of the usual Newton method applied to this problem

$$y^{n+1}(\varepsilon) = y^n(\varepsilon) - H_y(\varepsilon, y^n(\varepsilon))^{-1} H(\varepsilon, y^n(\varepsilon)). \tag{2.5.19}$$

At the moment we shall just make the assumptions necessary to obtain convergence results and afterwards consider additional conditions of the form (2.5.14). However we need Lipschitz conditions on the derivatives of G

$$\begin{aligned} &\text{a)}\ \|G_\lambda(\mu^1, z^1) - G_\lambda(\mu^2, z^2)\| \leq K_6\{(\|z^1\| + \|z^2\|)|\mu^1 - \mu^2| + \|z^1 - z^2\|\} \\ &\text{b)}\ \|G_x(\mu^1, z^1) - G_x(\mu^2, z^2)\| \leq K_7\{\|z^1 - z^2\| + |\mu^1 - \mu^2|\}. \end{aligned} \tag{2.5.20}$$

Theorem 2.9

For fixed ε , $|\varepsilon|$ sufficiently small, the Newton iteration (2.5.19), with starting approximation $y^0 = (\lambda^*, 0)$, converges to the bifurcating solution $y^*(\varepsilon)$.

Proof

We verify the conditions of Corollary 2.4. It has already been shown, in Theorem 2.7, that for sufficiently small $|\varepsilon| \neq 0$

(i) $H_y(\varepsilon, y^0)^{-1}$ exists with $\|H_y(\varepsilon, y^0)\| = O(|\varepsilon|^{-1})$

(ii) $\|H_y(\varepsilon, y^0)^{-1} H(\varepsilon, y^0)\| \leq K_1(\varepsilon)$ with $K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We therefore proceed to estimate a Lipschitz constant for

$$H_y(\varepsilon, y^0)^{-1}\{H_y(\varepsilon, z^1) - H_y(\varepsilon, z^2)\} \tag{2.5.21}$$

where $\|z^i - y^0\| < 2K_1(\varepsilon)$ and $z^i = (\mu^i, v^i)$ $i = 1, 2$.

$$\begin{aligned} H_y(\varepsilon, z^1) - H_y(\varepsilon, z^2) = {} & G_\lambda(\mu^1, \varepsilon(\phi^* + v^1)) - G_\lambda(\mu^2, \varepsilon(\phi^* + v^2)) \\ & + \varepsilon\{G_x(\mu^1, \varepsilon(\phi^* + v^1)) - G_x(\mu^2, \varepsilon(\phi^* + v^2))\} \end{aligned} \tag{2.5.22}$$

and thus, using (2.5.20) and (i) above,

$$\| H_y(\varepsilon, y^0)^{-1} \{ H_y(\varepsilon, z^1) - H_y(\varepsilon, z^2) \} \| \leq K_8 \| z^1 - z^2 \| . \qquad (2.5.23)$$

As $K_8 K_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ the conditions of Corollary 2.4 hold ∎

If, as in Corollary 2.8, we know more information about the operator G, the rate of convergence can be expressed in terms of ε.

<u>Corollary 2.10</u>

If $G(\lambda, x)$ can be written in the form

$$G(\lambda, x) = G_x(\lambda, 0)x + R(\lambda, x), \qquad (2.5.24)$$

where

$$\| R(\lambda, x) \| \leq K_9(\lambda) \| x \|^p \qquad (2.5.25)$$

with $p \geq 2$ and $K_9(\lambda)$ bounded, then the convergence rate for the iteration (2.5.19) is

$$\| y^n(\varepsilon) - y^*(\varepsilon) \| \leq (K_{10} |\varepsilon|^{p-1})^{2^n - 1} K_{11} |\varepsilon|^{p-1} / 2^{n-1} . \qquad (2.5.26)$$

<u>Proof</u>

Because $K_1(\varepsilon) = O(|\varepsilon|^{p-1})$ the result follows immediately from the last theorem and Corollary 2.4 ∎

Writing (2.5.26) in terms of λ and x we obtain

$$\begin{aligned} &\text{a)} \quad \| x^n(\varepsilon) - x^*(\varepsilon) \| \leq (K_{10} |\varepsilon|^{p-1})^{2^n - 1} K_{11} |\varepsilon|^p / 2^{n-1} \\ &\text{b)} \quad | \lambda^n(\varepsilon) - \lambda^*(\varepsilon) | \leq (K_{10} |\varepsilon|^{p-1})^{2^n - 1} K_{11} |\varepsilon|^{p-1} / 2^{n-1} . \end{aligned} \qquad (2.5.27)$$

We complete this section by considering the choice of N in the finite dimensional case. In general, it would appear that the following would be the easiest to implement. Let ϕ_s be the largest component, in modulus, of $\underset{\sim}{\phi}^*$ (now an n - vector). Let

$$l^*(x) = \underset{\sim}{e}_s^T \underset{\sim}{x} \tag{2.5.28}$$

and so N is the subspace of R^n annihilated by l^*, i.e. the subspace of n - vectors whose sth component is zero. In this case the difference between the nxn matrices $G_x(\lambda, \underset{\sim}{x})$ and $H_y(\varepsilon, \underset{\sim}{y}), x = \varepsilon\underset{\sim}{\phi}^* + \underset{\sim}{w}$ and $\underset{\sim}{y} = (\lambda, \underset{\sim}{w})$, only appears in the sth column, which, for $H_y(\varepsilon, \underset{\sim}{y})$, consists of $G_\lambda(\lambda, \underset{\sim}{x})$. Further aspects of this problem are brought out in later chapters.

2.6 Newton's Method with λ as Parameter

In this section we shall show that Newton iterations of the form (2.4.6) will converge under appropriate conditions. Although this method is not as general as that of section 2.5 it has several advantages. In applications λ is often an independent physical parameter and solutions are required for particular values of this parameter. Also, in finite dimensional approximations to continuous problems, the matrix $G_x(\lambda, x)$ is often of special form, e.g. banded, and we would like to keep this structure when solving sets of linear equations.

We assume that $G(\lambda, x)$ can be written in the form

$$G(\lambda, x) = G_x(\lambda, 0)x + C(\lambda)x + D(\lambda, x) \tag{2.6.1}$$

where, for each λ, $C(\lambda)$ is a bounded homogeneous operator of degree $p \geq 2$ and D is a higher-order operator satisfying

$$\begin{aligned} &a) \quad \|D(\lambda, x)\| \leq d_1(\lambda)\|x\|^{p+1} \\ &b) \quad \|D_x(\lambda, x)\| \leq d_2(\lambda)\|x\|^{p} \end{aligned} \tag{2.6.2}$$

with $d_i(\lambda)$ bounded $i = 1, 2$. For information about homogeneous and multi-linear operators see Krasnosel'skii [38], Vainberg and Trenogin [30], or Rall [24]. We shall use the result that if C is a homogeneous operator of degree p derived from a symmetric p-linear operator $\hat{C}$ then

$$C_x(x)y = p\,\hat{C}\underbrace{xx\ldots x}_{(p-1)}y \tag{2.6.3}$$

and so

$$C_x(x)x = pCx \tag{2.6.4}$$

The following Lipschitz conditions are also needed

$$\begin{aligned} &\text{a)}\quad \|G_{x\lambda}(\lambda,0) - G_{x\lambda}(\mu,0)\| = O(|\lambda-\mu|) \\ &\text{b)}\quad \|C(\lambda) - C(\mu)\| = O(|\lambda-\mu|) \;. \end{aligned} \tag{2.6.5}$$

Now, if the Liapunov-Schmidt analysis of section 2.2 is carried out, (2.2.11) becomes

$$a\,\delta\lambda + b\varepsilon^{p-1} + O\left(|\delta\lambda|^2 + |\varepsilon|^{p-1}(|\delta\lambda|+|\varepsilon|)\right) = 0 , \tag{2.6.6}$$

where

$$b = \psi^*(C(\lambda^*)\phi^*) \;. \tag{2.6.7}$$

In this section we shall assume that $b \neq 0$ and consider some contrary cases in section 2.7. Thus (2.6.6) may be solved for ε in terms of $\delta\lambda$ to give

$$\varepsilon(\delta\lambda) = (-a\,\delta\lambda/b)^{1/p-1} + O\left(|\delta\lambda|^{2/p-1}\right) . \tag{2.6.8}$$

Of course, if p is odd, (2.6.6) shows that $\delta\lambda$ is of one sign for ε sufficiently small and we must keep to this sign when using (2.6.8). In this case taking positive or negative roots will give the two values of ε for a given $\delta\lambda$. Using (2.6.8) we may substitute for ε in the expression derived for $w(\delta\lambda,\varepsilon)$ in section 2.2, and thus obtain the bifucating solutions in terms of $\delta\lambda$ i.e. $x^*(\delta\lambda)$. Always remembering that for odd p we must consider each half-branch, with positive or negative root in (2.6.8), separately.

Now the first term of (2.6.8) can be used as a starting value for Newton's method and we begin by proving the following theorem for the modified iteration

$$x^{n+1}(\delta\lambda) = x^n(\delta\lambda) - G_x(\lambda^*+\delta\lambda, x^0(\delta\lambda))^{-1} G(\lambda^*+\delta\lambda, x^n(\delta\lambda)) \;. \tag{2.6.9}$$

Theorem 2.11

For fixed $\delta\lambda$ sufficiently small the Newton sequence (2.6.9), with starting approximation

$$x^{\circ}(\delta\lambda) = (-a\,\delta\lambda/b)^{1/p-1}\,\phi^{*}\,, \tag{2.6.10}$$

will converge to the bifurcating solution $x^{*}(\delta\lambda)$, provided that, if p is odd, $\delta\lambda$ is restricted to the sign of $(-a/b)$.

Proof

We verify the conditions of Theorem 2.6. Firstly

$$\begin{aligned} G_x(\lambda, x^{\circ}(\delta\lambda)) &= G_x(\lambda,0) + C_x(\lambda, x^{\circ}(\delta\lambda)) + D_x(\lambda, x^{\circ}(\delta\lambda)) \\ &= G_x(\lambda^{*},0) + \delta\lambda\, G_{x\lambda}(\lambda^{*},0) + C_x(\lambda^{*}, x^{\circ}(\delta\lambda)) \\ &\quad + \{G_x(\lambda,0) - G_x(\lambda^{*},0) - \delta\lambda\, G_{x\lambda}(\lambda^{*},0)\} \\ &\quad + \{C_x(\lambda, x^{\circ}(\delta\lambda)) - C_x(\lambda^{*}, x^{\circ}(\delta\lambda))\} \\ &\quad + D_x(\lambda, x^{\circ}(\delta\lambda))\,. \end{aligned} \tag{2.6.11}$$

By definition of $x^{\circ}(\delta\lambda)$, and using (2.1.9) and (2.6.7) in Lemma 2.1, it follows that

$$G_x(\lambda^{*},0) + \delta\lambda\, G_{x\lambda}(\lambda^{*},0) + C_x(\lambda^{*}, x^{\circ}(\delta\lambda)) \tag{2.6.12}$$

is an invertible operator for sufficiently small $|\delta\lambda| \neq 0$. Additionally if $\|x\| = 1$,

$$\|\{G_x(\lambda^{*},0) + \delta\lambda\, G_{x\lambda}(\lambda^{*},0) + C_x(\lambda^{*}, x^{\circ}(\delta\lambda))\}^{-1} x\| = O(1) \tag{2.6.13}$$

if $x \in \mathcal{R}\{G_x(\lambda^{*},0)\}$ and

$$\|\{G_x(\lambda^{*},0) + \delta\lambda\, G_{x\lambda}(\lambda^{*},0) + C_x(\lambda^{*}, x^{\circ}(\delta\lambda))\}^{-1} x\| = O(|\delta\lambda|^{-1}) \tag{2.6.14}$$

if $x \notin \mathcal{R}\{G_x(\lambda^{*},0)\}$. The other terms on the right-hand side of (2.6.11) are of higher-order in $\delta\lambda$ and so we may use the Banach perturbation theorem to conclude that $G_x(\lambda, x^{\circ}(\delta\lambda))^{-1}$ exists and satisfies bounds of the form (2.6.13-14), i.e. if $\|x\| = 1$

$$\begin{aligned} &\text{a)}\quad \|G_x(\lambda, x^{\circ}(\delta\lambda))^{-1} x\| = O(1) && x \in \mathcal{R}\{G_x(\lambda^{*},0)\} \\ &\text{b)}\quad \|G_x(\lambda, x^{\circ}(\delta\lambda))^{-1} x\| = O(|\delta\lambda|^{-1}) && x \notin \mathcal{R}\{G_x(\lambda^{*},0)\}\,. \end{aligned} \tag{2.6.15}$$

Secondly,

$$G(\lambda, x^\circ(\delta\lambda)) = G_x(\lambda,0)x^\circ(\delta\lambda) + C(\lambda)x^\circ(\delta\lambda) + D(\lambda, x^\circ(\delta\lambda)) \tag{2.6.16}$$

$$= G_x(\lambda,0)x^\circ(\delta\lambda) + G_x(\lambda^*,0)x^\circ(\delta\lambda) + G_x$$

$$+ \delta\lambda\, G_{x\lambda}(\lambda^*,0)\,x^\circ(\delta\lambda) + C(\lambda^*)x^\circ(\delta\lambda)$$

$$+ \{C(\lambda) - C(\lambda^*)\}\, x^\circ(\delta\lambda) + D(\lambda, x^\circ(\delta\lambda))$$

and so, by definition of $x^\circ(\delta\lambda)$,

a) $\|G(\lambda, x^\circ(\delta\lambda))\| = O(|\delta\lambda|^{p/p-1})$

but b) $|\psi^*(G(\lambda, x^\circ(\delta\lambda)))| = O(|\delta\lambda|^{(p+1)/p-1})$. (2.6.17)

Because $\mathcal{R}\{G_x(\lambda^*,0)\}$ is the subspace of X annihilated by ψ^* we may combine (2.6.15) and (2.6.17) to obtain

$$\|G_x(\lambda, x^\circ(\delta\lambda))^{-1} G(\lambda, x^\circ(\delta\lambda))\| \leqslant K_1(\delta\lambda) \tag{2.6.18}$$

where $K_1(\delta\lambda) = O(|\delta\lambda|^{2/p-1})$. Thirdly, if z^i $i=1,2$ satisfy $\|z^i - x^\circ(\delta\lambda)\| <$ $2K_1(\delta\lambda)$, then

$$G_x(\lambda, x^\circ(\delta\lambda))(z^1 - z^2) - (G(\lambda, z^1) - G(\lambda, z^2)) \tag{2.6.19}$$

$$= C_x(\lambda, x^\circ(\delta\lambda))(z^1 - z^2) - (C(\lambda)z^1 - C(\lambda)z^2)$$

$$+ D_x(\lambda, x^\circ(\delta\lambda))(z^1 - z^2) - (D(\lambda, z^1) - D(\lambda, z^2))$$

and, using (2.6.2),(2.6.15) and the homogeneity of C, it follows that

$$G_x(\lambda, x^\circ(\delta\lambda))^{-1}\{G_x(\lambda, x^\circ(\delta\lambda))(z^1 - z^2) - (G(\lambda, z^1) - G(\lambda, z^2))\} \tag{2.6.20}$$

$$\leqslant K_2(\delta\lambda)\, \|z^1 - z^2\|$$

where $K_2(\delta\lambda) = O(|\delta\lambda|^{1/p-1})$.

Thus the conditions of Theorem 2.6 hold and the theorem is proved. The convergence rate is given by

$$\|x^n(\delta\lambda) - x^*(\delta\lambda)\| \leqslant (K_3|\delta\lambda|^{1/p-1})^n K_4 |\delta\lambda|^{2/p-1}. \tag{2.6.21}$$

Comparing (2.6.21) with Corollary 2.8, using ϵ as the parameter, we see that the convergence rate is the same for $p = 2$ but slower for $p > 2$.

If instead of the usual modified Newton method, we replace $G_x(\lambda, x^\circ(\delta\lambda))$ by an approximation, such as

$$\text{a)}\quad G_x(\lambda,0)+C_x(\lambda,x^\circ(\delta\lambda))$$

$$\text{or}\quad \text{b)}\quad G_x(\lambda^*,0)+\delta\lambda G_{x\lambda}(\lambda^*,0)+C_x(\lambda^*,x^\circ(\delta\lambda)) \tag{2.6.22}$$

it is easy to see that a modification of Theorem 2.11 holds, with the same convergence rate in terms of $\delta\lambda$. In this case it is not necessary for D to be differentiable with respect to x , a Lipschitz condition of the form

$$\|D(\lambda,z^1)-D(\lambda,z^2)\| \leq d_3(\lambda)\left(\|z^1\|^p+\|z^2\|^p\right)\|z^1-z^2\| \tag{2.6.23}$$

with $d_3(\lambda)$ bounded, is sufficient.

We now proceed to consider the full Newton method

$$x^{n+1}(\delta\lambda)=x^n(\delta\lambda)-G_x(\lambda^*+\delta\lambda,x^n(\delta\lambda))^{-1}G(\lambda^*+\delta\lambda,x^n(\delta\lambda)), \tag{2.6.24}$$

and for this we require a Lipschitz condition on D_x

$$\|D_x(\lambda,z^1)-D_x(\lambda,z^2)\| \leq d_4(\lambda)\left(\|z^1\|^{p-2}+\|z^2\|^{p-2}\right)\|z^1-z^2\|. \tag{2.6.25}$$

Theorem 2.12

For fixed $\delta\lambda$ sufficiently small the Newton iteration (2.6.24) will converge to the bifurcating solution $x^*(\delta\lambda)$ when the starting approximation is given by (2.6.10), with $\delta\lambda$ similarly restricted if p is odd .

Proof

We check the conditions of Corollary 2.4. From the last theorem, for sufficiently small $\delta\lambda \neq 0$,

(i) $G_x(\lambda,x^\circ(\delta\lambda))^{-1}$ exists with $\|G_x(\lambda,x^\circ(\delta\lambda))^{-1}\| = O(|\delta\lambda|^{-1})$

(ii) $\|G_x(\lambda,x^\circ(\delta\lambda))^{-1}G(\lambda,x^\circ(\delta\lambda))\| \leq K_1(\delta\lambda)$ with $K_1(\delta\lambda)=O(|\delta\lambda|^{2/p-1})$.

Now we estimate a Lipschitz constant for

$$G_x(\lambda,x^\circ(\delta\lambda))^{-1}\left\{G_x(\lambda,z^1)-G_x(\lambda,z^2)\right\} \tag{2.6.26}$$

where $\|z^i-x^\circ(\delta\lambda)\| < 2K_1(\delta\lambda)\quad i=1,2$.

$$G_x(\lambda, z^1) - G_x(\lambda, z^2) = C_x(\lambda, z^1) - C_x(\lambda, z^2) + D_x(\lambda, z^1) - D_x(\lambda, z^2) \qquad (2.6.27)$$

and using the homogeneity of $C(\lambda)$ with (2.6.25) and (i) above gives

$$\|G_x(\lambda, x^*(\delta\lambda))^{-1}\{G_x(\lambda, z^1) - G_x(\lambda, z^2)\}\| \leq K_5(\delta\lambda)\|z^1 - z^2\| \qquad (2.6.28)$$

with $K_5(\delta\lambda) = O(|\delta\lambda|^{-1/p-1})$. Thus Corollary 2.4 can be applied and we obtain the convergence rate

$$\|x^n(\delta\lambda) - x^*(\delta\lambda)\| \leq (K_6|\delta\lambda|^{1/p-1})^{2^n-1} K_7|\delta\lambda|^{2/p-1}/2^{n-1} \blacksquare \qquad (2.6.29)$$

This theorem is illustrated by numerical results in section 2.8.

2.7 Newton's Method with λ as Parameter - Degenerate Case

In this section we consider two of the more commonly occurring situations when b defined by (2.6.7), is zero. Of course, it is impossible to use λ as a parameter for all problems because the bifurcating solutions may only exist for $\lambda = \lambda^*$. It is assumed that G can be written

$$G(\lambda, x) = G_x(\lambda, 0) + \sum_{k=p}^{m} C_k(\lambda)x + D(\lambda, x), \qquad (2.7.1)$$

where each $C_k(\lambda)$ is a homogeneous operator of degree k, see (2.6.1), and D is a higher-order operator satisfying (2.6.2) with p replaced by m. We also assume that $G_{x\lambda}(\lambda, 0)$ satisfies (2.6.5a) and each $C_k(\lambda)$ satisfies (2.6.5b). Attention is restricted to $m \leq 2p-1$ and we consider the two sub-cases $m < 2p-1$ and $m = 2p-1$ separately.

2.7.1 $m < 2p-1$

By analogy with (2.6.7) it is assumed that

$$\begin{aligned} &a) \quad \psi^*(C_k(\lambda^*)\phi^*) = 0 \qquad p \leq k < m \\ \text{but} \quad &b) \quad b_m = \psi^*(C_m(\lambda^*)\phi^*) \neq 0 \ . \end{aligned} \qquad (2.7.2)$$

The Liapunov-Schmidt analysis of section (2.2) can be repeated so that (2.2.11) becomes

$$a\,\delta\lambda + b_m\,\varepsilon^{m-1} + O\left(|\delta\lambda|^2 + |\varepsilon|^{m-1}(|\delta\lambda| + |\varepsilon|)\right) = 0\,, \tag{2.7.3}$$

which can be solved for ε in terms of $\delta\lambda$ to give

$$\varepsilon(\delta\lambda) = \left(-a\,\delta\lambda/b_m\right)^{1/m-1} + O\left(|\delta\lambda|^{2/m-1}\right). \tag{2.7.4}$$

As in the previous section, see (2.6.8), if m is odd the sign of $\delta\lambda$ must be restricted and positive and negative roots taken in (2.7.4). This expression for ε is then substituted into $w(\delta\lambda,\varepsilon)$ of section (2.2) to give the bifurcating solutions in terms of $\delta\lambda$, $x^*(\delta\lambda) = \varepsilon(\delta\lambda)\phi^* + w(\delta\lambda, \varepsilon(\delta\lambda))$. The first term on the right-hand side of (2.7.4) can now be used as an initial approximation for $x^*(\delta\lambda)$.

Before proving the convergence of the modified and full Newton's methods we need to introduce two new norms on X,

$$\begin{aligned} &\text{a)} \quad \|x\|_1 = \|\ell^*(x)\phi^* + |\delta\lambda|^{(p-m)/m-1}\, Qx\| \\ &\text{b)} \quad \|x\|_2 = \|\psi^*(x)z^* + |\delta\lambda|^{(p-m)/m-1}\, Px\|\,, \end{aligned} \tag{2.7.5}$$

where ℓ^* and z^* are defined in (2.1.3) and P and Q in (2.1.4). The effect of (2.7.5a) is to multiply the component of x in N by $|\delta\lambda|^{(p-m)/m-1}$ and similarly (2.7.5b) for the component of x in $\mathcal{R}\{G_x(\lambda^*,0)\}$. Throughout this section we denote X normed by $\|\ \|_1$ by Z_1 and X normed by $\|\ \|_2$ by Z_2. The norms of linear operators from Z_1 to Z_2 will be denoted by $\|\ \|_{12}$ and similarly those from Z_2 to Z_1 by $\|\ \|_{21}$.

Now we can prove the convergence of Newton's method and we commence with the modified iteration (2.6.9).

<u>Theorem 2.13</u>

For fixed $\delta\lambda$ sufficiently small the Newton iterates (2.6.9), with starting approximation

$$x^0(\delta\lambda) = \left(-a\,\delta\lambda/b_m\right)^{1/m-1}\phi^*\,, \tag{2.7.6}$$

converge to the bifurcating solution $x^*(\delta\lambda)$; provided that, if m is odd

$\delta\lambda$ is restricted to the sign of $(-a/b_m)$.

Proof

We proceed as in Theorem 2.11, but using the new norms (2.7.5). In the interest of notational clarity we refer to the derivative of $C_k(\lambda)x$ with respect to x at y as $C_{k-1,1}(\lambda,y)$.

$$\begin{aligned} G_x(\lambda,x^\circ(\delta\lambda)) &= G_x(\lambda,0)+\sum_{k=p}^{m}C_{k-1,1}(\lambda,x^\circ(\delta\lambda)) \quad + D_x(\lambda,x^\circ(\delta\lambda)) \qquad (2.7.7)\\ &= G_x(\lambda^*,0)+\delta\lambda G_{x\lambda}(\lambda^*,0)+\sum_{k=p}^{m}C_{k-1,1}(\lambda^*,x^\circ(\delta\lambda))\\ &\quad + G_x(\lambda,0)-G_x(\lambda^*,0)-\delta\lambda G_{x\lambda}(\lambda^*,0)\\ &\quad + \sum_{k=p}^{m}\{C_{k-1,1}(\lambda,x^\circ(\delta\lambda))-C_{k-1,1}(\lambda^*,x^\circ(\delta\lambda))\}\\ &\quad + D_x(\lambda,x^\circ(\delta\lambda)) . \end{aligned}$$

Using the direct sums (2.1.5) we can express $G_x(\lambda,x^\circ(\delta\lambda))$ as a decomposable operator (see Krasnosel'skii [38], p. 402) with matrix representation

$$G_x(\lambda,x^\circ(\delta\lambda)) = \begin{pmatrix} L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix} \qquad (2.7.8)$$

where $L_{11}:\{\phi^*\}\to\{z^*\}$, $L_{12}:N\to\{z^*\}$ etc.. The following bounds can also be obtained from (2.7.2)

$$\begin{aligned} &\|L_{11}\|_{12} = O(|\delta\lambda|) && \|L_{12}\|_{12} = O(|\delta\lambda|)\\ &\|L_{21}\|_{12} = O(|\delta\lambda|^{(2p-1-m)/m-1)}) && \|L_{22}\|_{12} = O(1) \end{aligned} \qquad (2.7.9)$$

and L_{11} and L_{12} are invertible for sufficiently small $\delta\lambda\neq 0$ with

$$\|L_{11}^{-1}\|_{21} = O(|\delta\lambda|^{-1}) \qquad \|L_{22}^{-1}\|_{21} = O(1) . \qquad (2.7.10)$$

Now, using simple algebra and writing $J=L_{22}^{-1}L_{21}K^{-1}L_{12}L_{22}^{-1}$ with $K=L_{11}-L_{12}L_{22}^{-1}L_{21}$, which is invertible by the Banach perturbation theorem for sufficiently small $\delta\lambda\neq 0$, we can show that $G_x(\lambda,x^\circ(\delta\lambda))^{-1}$ exists with representation

$$\begin{pmatrix} K^{-1} & -K^{-1}L_{12}L_{22}^{-1}\\ -L_{22}^{-1}L_{21}K^{-1} & L_{22}^{-1}+J\end{pmatrix} . \qquad (2.7.11)$$

If $\|x_2\|_2=1$ then (2.7.11) also shows that

a) $\|G_x(\lambda,x^\circ(\delta\lambda))^{-1}x\|_1 = O(1) \quad x\in\mathcal{R}\{G_x(\lambda^*,0)\}$

b) $\|G_x(\lambda,x^\circ(\delta\lambda))^{-1}x\|_1 = O(|\delta\lambda|^{-1}) \quad x\notin\mathcal{R}\{G_x(\lambda^*,0)\}$. (2.7.12)

Continuing to verify the conditions of Theorem 2.6 we have

$$\begin{aligned} G(\lambda, x^\circ(\delta\lambda)) &= G_x(\lambda,0)x^\circ(\delta\lambda) + \sum_{k=p}^{m} C_k(\lambda) x^\circ(\delta\lambda)) \quad + D(\lambda, x^\circ(\delta\lambda)) \\ &= G_x(\lambda,0)x^\circ(\delta\lambda) - G_x(\lambda^*,0)x^\circ(\delta\lambda) - \delta\lambda G_{x\lambda}(\lambda^*,0)x^\circ(\delta\lambda) \\ &\quad + \delta\lambda G_{x\lambda}(\lambda^*,0)x^\circ(\delta\lambda) + \sum_{k=p}^{m} C_k(\lambda^*) x^\circ(\delta\lambda) \\ &\quad + \sum_{k=p}^{m} (C_k(\lambda) - C_k(\lambda^*)) x^\circ(\delta\lambda) \quad + D(\lambda, x^\circ(\delta\lambda)) \end{aligned} \tag{2.7.13}$$

and thus, using (2.7.2),

$$\begin{aligned} &\text{a)} \quad \| G(\lambda, x^\circ(\delta\lambda))\|_2 = O(|\delta\lambda|^{(2p-m)/m-1}) \\ \text{but} \quad &\text{b)} \quad |\psi^*(G(\lambda, x^\circ(\delta\lambda)))| = O(|\delta\lambda|^{(m+1)/m-1}) \; . \end{aligned} \tag{2.7.14}$$

Combining (2.7.12) and (2.7.14) gives

$$\| G_x(\lambda, x^\circ(\delta\lambda))^{-1} G(\lambda, x^\circ(\delta\lambda))\| \leqslant K_1(\delta\lambda) \tag{2.7.15}$$

where $K_1(\delta\lambda) = O(|\delta\lambda|^{2/m-1})$.

Finally, if y^i $i=1,2$ satisfy $\| y^i - x^\circ(\delta\lambda)\|_1 < 2K_1(\delta\lambda)$,

$$\begin{aligned} G_x(\lambda, x^\circ(\delta\lambda))(y^1-y^2) &- (G(\lambda,y^1) - G(\lambda,y^2)) \\ &= \sum_{k=p}^{m} C_{k-1,1}(\lambda^*, x^\circ(\delta\lambda))(y^1-y^2) - (C_k(\lambda^*)y^1 - C_k(\lambda^*)y^2) \\ &\quad + \sum_{k=p}^{m} \{ C_{k-1,1}(\lambda, x^\circ(\delta\lambda)) - C_{k-1,1}(\lambda^*, x^\circ(\delta\lambda))\} (y^1-y^2) \\ &\quad + \sum_{k=p}^{m} \{ C_k(\lambda)y^1 - C_k(\lambda^*)y^1 + C_k(\lambda)y^2 - C_k(\lambda^*)y^2 \} \\ &\quad + D_x(\lambda, x^\circ(\delta\lambda))(y^1-y^2) - (D(\lambda,y^1) - D(\lambda,y^2)) \end{aligned} \tag{2.7.16}$$

and, using (2.7.2) and the homogeneity of the C_k, gives

$$\begin{aligned} &\text{a)} \quad \| G_x(\lambda, x^\circ(\delta\lambda))(y^1-y^2) - (G(\lambda,y^1) - G(\lambda,y^2))\|_2 \\ &\qquad\qquad \leqslant K_2(\delta\lambda)\|y^1-y^2\|_1 \\ \text{but} \quad &\text{b)} \quad |\psi^*(G_x(\lambda, x^\circ(\delta\lambda))(y^1-y^2) - (G(\lambda,y^1) - G(\lambda,y^2)))| \\ &\qquad\qquad \leqslant K_3(\delta\lambda)\|y^1-y^2\|_1 \end{aligned} \tag{2.7.17}$$

where $K_2(\delta\lambda) = O(|\delta\lambda|^{(2p-m)/m-1})$ and $K_3(\delta\lambda) = O(|\delta\lambda|^{m/m-1})$.

Combining (2.7.12) and (2.7.17) we obtain

$$\begin{aligned} \| G_x(\lambda, x^\circ(\delta\lambda))^{-1} \{ G_x(\lambda, x^\circ(\delta\lambda))(y^1-y^2) - (G(\lambda,y^1) - G(\lambda,y^2))\} \|_1 \\ \leqslant K_4(\delta\lambda)\|y^1-y^2\|_1 \end{aligned} \tag{2.7.18}$$

where $K_4(\delta\lambda) = O(|\delta\lambda|^{1/m-1})$.

Thus the conditions of Theorem 2.6. hold and the theorem is proved, with a convergence rate for the iterates given by

$$\|x^n(\delta\lambda)-x^*(\delta\lambda)\|_1 \leq \left(K_5|\delta\lambda|^{1/m-1}\right)^n K_6|\delta\lambda|^{2/m-1} . \qquad (2.7.19)$$

Using (2.7.5a) we can also express the convergence rate in the original norm ∎

As in the previous section we can approximate $G_x(\lambda, x^0(\delta\lambda))$ in the modified Newton method and still retain the same order of convergence in powers of $\delta\lambda$, i.e. using

$$\text{a)} \quad G_x(\lambda,0)+\sum_{k=p}^{m} C_{k-1,1}(\lambda,x^0(\delta\lambda))$$

$$\text{or} \quad \text{b)} \quad G_x(\lambda^*,0)+\delta\lambda\, G_{x\lambda}(\lambda^*,0)+\sum_{k=p}^{m} C_{k-1,1}(\lambda^*,x^0(\delta\lambda)) . \qquad (2.7.20)$$

We now consider the full Newton iteration (2.6.24) for which we need to assume a Lipschitz condition on D_x of the form (2.6.25), with p replaced by m.

Theorem 2.14

For fixed $\delta\lambda$ sufficiently small the Newton sequence (2.6.24) will converge to the bifurcating solution $x^*(\delta\lambda)$ when the starting approximation is given by (2.7.6); provided that $\delta\lambda$ is similarly restricted if m is odd .

Proof

Conditions (i) and (vi) of Corollary 2.4 have already been verified in Theorem 2.13, with $\eta(\delta\lambda)=K_1(\delta\lambda)=O(|\delta\lambda|^{2/m-1})$. We now show that condition (iii) holds.

If $\|y^i-x^0(\delta\lambda)\|_1 < 2K_1(\delta\lambda)$ $i=1,2$ then

$$G_x(\lambda,y^1)-G_x(\lambda,y^2)=\sum_{k=p}^{m}\left\{C_{k-1,1}(\lambda^*,y^1)-C_{k-1,1}(\lambda^*,y^2)\right\} \qquad (2.7.21)$$

$$+\sum_{k=p}^{m}\{C_{k-1,1}(\lambda,y^1)-C_{k-1,1}(\lambda^*,y^1)\}$$
$$+\sum_{k=p}^{m}\{C_{k-1,1}(\lambda,y^2)-C_{k-1,1}(\lambda^*,y^2)\}$$
$$+D_x(\lambda,y^1)-D_x(\lambda,y^2)$$

and, using (2.7.2) with the homogeneity of the C_k and the Lipschitz condition on D_x, we obtain

$$\text{a)}\quad \|G_x(\lambda,y^1)-G_x(\lambda,y^2)\|_{12}\leqslant K_7(\delta\lambda)\,\|y^1-y^2\|_1 \tag{2.7.22}$$

but
$$\text{b)}\quad |\psi^*(G_x(\lambda,y^1)-G_x(\lambda,y^2))|\leqslant K_8(\delta\lambda)\,\|y^1-y^2\|_1$$

where $K_7(\delta\lambda)=O(|\delta\lambda|^{(2p-2-m)/m-1})$ and $K_8(\delta\lambda)=O(|\delta\lambda|^{(m-2)/m-1})$.

Combining (2.7.12) and (2.7.22) gives

$$\|G_x(\lambda,x^0(\delta\lambda))^{-1}\{G_x(\lambda,y^1)-G_x(\lambda,y^2)\}\|_1\leqslant K_9(\delta\lambda)\,\|y^1-y^2\|_1 \tag{2.7.23}$$

where $K_9(\delta\lambda)=O(|\delta\lambda|^{-1/m-1})$.

We can now apply Corollary 2.4 to prove the theorem and give the convergence rate

$$\|x^n(\delta\lambda)-x^*(\delta\lambda)\|_1\leqslant\left(K_{10}|\delta\lambda|^{1/m-1}\right)^{2^n-1}K_{11}|\delta\lambda|^{2/m-1}/2^{n-1} \tag{2.7.24}$$

which can be expressed in terms of the original norm by using (2.7.5a)∎

2.7.2 $m=2p-1$

The conditions, analogous to (2.7.2), which we impose on (2.7.1) in this section are

$$\text{a)}\quad \psi^*(C_k(\lambda^*)\phi^*)=0 \qquad p\leqslant k<m \tag{2.7.25}$$
$$\text{b)}\quad b_m=\psi^*(C_m(\lambda^*)\phi^*+C_{p-1,1}(\lambda^*,\phi^*)w^*)\neq 0,$$

where $w^*=-MC_p(\lambda^*)\phi^*$ with M defined in section 2.1. The significance of these assumptions becomes clear when the Liapunov-Schmidt analysis of section 2.2 is carried out. The function $w(\delta\lambda,\varepsilon)$ obtained by solving (2.2.5), is of the form

$$w(\delta\lambda,\varepsilon)=\varepsilon^p w^*+O(|\varepsilon|^p(|\delta\lambda|+|\varepsilon|)) \tag{2.7.26}$$

and, when this is inserted into (2.2.4b), the counterpart to (2.2.11) becomes

$$a\,\delta\lambda + b_m\,\varepsilon^{2p-2} + O\left(|\delta\lambda|^2 + |\varepsilon|^{2p-2}(|\varepsilon| + |\delta\lambda|)\right) = 0 \ . \qquad (2.7.27)$$

With condition (2.7.25b) this can be solved for ε in terms of $\delta\lambda$ to give

$$\varepsilon(\delta\lambda) = \left(-a\,\delta\lambda/b_m\right)^{1/2p-2} + O\left(|\delta\lambda|^{1/p-1}\right) \ . \qquad (2.7.28)$$

Here, of course, we must restrict $\delta\lambda$ to the sign of $(-a/b_m)$ and then take positive and negative roots in order to obtain the bifurcating solutions for positive and negative ε.

The proofs of the convergence of the modified and full Newton's methods are virtually the same as for $m<2p-1$ and so we merely state the theorems below. The two differences are

(i) the norms to be used instead of (2.7.5) are

$$\begin{aligned} &\text{a)}\ \ \|x\|_1 = \|L^*(x)\phi^* + |\delta\lambda|^{-1/2}\,Qx\| \\ &\text{b)}\ \ \|x\|_2 = \|\psi^*(x)z^* + |\delta\lambda|^{-1/2}\,Px\| \end{aligned} \qquad (2.7.29)$$

(ii) the starting approximation used is

$$x^0(\delta\lambda) = (-a\,\delta\lambda/b_m)^{1/2p-2}\,\phi^* + (-a\,\delta\lambda/b_m)^{p/2p-2}\,w^* \ . \qquad (2.7.30)$$

Theorem 2.15

For fixed $\delta\lambda$ sufficiently small and of the same sign as $(-a/b_m)$, the modified Newton iteration (2.6.9), with starting approximation given by (2.7.30), converges to the bifurcating solution $x^*(\delta\lambda)$ at a rate given by

$$\|x^n(\delta\lambda) - x^*(\delta\lambda)\|_1 \le \left(K_{12}|\delta\lambda|^{1/2p-2}\right)^n K_{13}\,|\delta\lambda|^{1/p-1} . \qquad (2.7.31)$$

For the full Newton method we need the extra assumption on D_x given by (2.6.25), with p replaced by $m=2p-1$.

Theorem 2.16

For fixed $\delta\lambda$ sufficiently small and of the same sign as $(-a/b_m)$, the Newton iteration (2.6.24), with starting value given by (2.7.30), converges to the bifurcating solution $x^*(\delta\lambda)$ at a rate given by

$$\|x^n(\delta\lambda) - x^*(\delta\lambda)\|_1 \leq \left(K_{14}|\delta\lambda|^{1/2p-2}\right)^{2^n-1} K_{15}|\delta\lambda|^{1/p-1}/2^{n-1}. \quad (2.7.32)$$

Numerical results for this theorem are given in the next section.

2.8 Numerical Results

We restrict ourselves to giving numerical results for continuation with λ , as in sections 2.6 and 2.7.

The problem considered is the two point boundary-value problem

$$\begin{aligned} &\text{a)} \quad y'' - \lambda(y + C(y) + D(y)) = 0 \\ &\text{b)} \quad y(0) = y(1) = 0 \,. \end{aligned} \tag{2.8.1}$$

The eigen-values and eigen-functions of the linearised problem are well-known to be

$$\begin{aligned} &\text{a)} \quad \lambda^* = -k^2\pi^2 \\ &\text{b)} \quad \phi^* = \sin(k\pi x) \,. \end{aligned} \qquad k = 1, 2, \ldots \tag{2.8.2}$$

The difference method used was

$$(y_{i+1} - 2y_i + y_{i-1})/h^2 - \lambda(y_i + C(y_i) + D(y_i)) = 0 \tag{2.8.3}$$

for $i = 1, \ldots, (n-1)$ with $h = 1/n$ and $y_0 = y_n = 0$. The eigen-values and eigen-vectors of the linearisation of the discretized problem are

$$\begin{aligned} &\text{a)} \quad \lambda_h^* = -4n^2 \sin^2(k\pi/(2n)) \\ &\text{b)} \quad (\phi_h^*)_i = \sin(k\pi i/n) \qquad i = 0, \ldots, n \end{aligned} \tag{2.8.4}$$

for $k = 1, \ldots, n-1$. The convergence of the solutions of such approximations to those of the continuous problem was treated in Weiss [50].

Three examples are given, with different λ, C and D, to illustrate the previous theory. All iterations show quadratic convergence and were terminated when correct to 10 decimal places.

Example 1

Take $C(y)=y^2$ and $D(y)=y^3$ and consider the branch from $\lambda_h^* = -4n^2 \sin^2(\pi/(2n))$. By direct verification (2.6.7) holds with p = 2. We take n = 40 and the results are given in Table 1. The values quoted are the n/2 components of the vectors i.e. approximations to y(0.5).

Table 1

	$\delta\lambda = 0.1$		$\delta\lambda = -0.5$	
r	$(\underline{y})_{n/2}$	$\|\delta \underline{y}\|_\infty$	$(\underline{y})_{n/2}$	$\|\delta \underline{y}\|_\infty$
0	0.119427529 E-1		-0.597137643 E-1	
		0.58 E-5		0.29 E-3
1	0.119408133 E-1		-0.599759741 E-1	
		0.21 E-8		0.17 E-5
2	0.119408112 E-1		-0.599743183 E-1	
				0.43 E-10
3			-0.599743182 E-1	

Example 2

Take $C(y)=y^4$ and $D(y)=y^5$ and consider the branch from $\lambda_h^* = -4n^2 \sin^2(\pi/n)$ i.e. $k=2$ in (2.8.4a). In this case $\underline{\phi}_h^{*T} C(\underline{\phi}_h^*)=0$ and $m<2p-1$ and so section 2.7.1 applies. The bifurcating solution curve only exists for positive $\delta\lambda$ and we give the positive values of $(\underline{y})_{n/4}$, the approximation to y(0.25), in Table 2, taking n = 40.

Table 2

r	$\delta\lambda = 0.1$ $(\underset{\sim}{y})_{n/4}$	$\|\delta \underset{\sim}{y}\|_\infty$	$\delta\lambda = 0.5$ $(\underset{\sim}{y})_{n/4}$	$\|\delta \underset{\sim}{y}\|_\infty$
0	0.2524430118		0.3774903501	
		0.73 E-2		0.33 E-1
1	0.2572273206		0.3943410274	
		0.45 E-4		0.41 E-3
2	0.2572656246		0.3943979335	
		0.78 E-8		0.19 E-6
3	0.2572656167		0.3943980651	

Example 3

Take $C(\underset{\sim}{y}) = \underset{\sim}{y}^2$ and $D(\underset{\sim}{y}) = \underset{\sim}{y}^3$, as in Example 1, but consider the branch from the second eigenvalue, $\lambda_h^* = -4n^2 \sin^2(\pi/n)$. In this case $\underset{\sim}{\phi}_h^{*T} C(\underset{\sim}{\phi}_h^*) = 0$ and $m = 2p-1$ and the analysis of section 2.7.2 applies. The bifurcating solution curve only exists for negative $\delta\lambda$ and we give the negative values of $(\underset{\sim}{y})_{n/4}$ in Table 3, taking $n = 40$.

The reason for choosing relatively small values of $\delta\lambda$ is that there is a turning point on the branch near $\delta\lambda = 0.06$. This example is used to illustrate the theory of chapter 3 in section 3.7.

Table 3

	$\delta\lambda = -0.01$		$\delta\lambda = -0.05$	
r	$(\underset{\sim}{y})_{n/4}$	$\|\delta \underset{\sim}{y}\|_\infty$	$(\underset{\sim}{y})_{n/4}$	$\|\delta \underset{\sim}{y}\|_\infty$
0	-0.566802065 E-1		-0.1295824299	
		0.13 E-2		0.27 E-1
1	-0.579833692 E-1		-0.1550539853	
		0.21 E-4		0.68 E-3
2	-0.579624591 E-1		-0.1551589251	
		0.42 E-7		0.12 E-5
3	-0.579655014 E-1		-0.1551600636	
				0.27 E-9
4			-0.1551600633	

CHAPTER 3

TURNING POINTS

3.1 Introduction

In this chapter we are again concerned with the problem of computing solutions of (1.1) near a singularity, and we make the same smoothness assumptions on G. However, in contrast to chapter 2, we are interested in singularities of $G_x(\lambda, x)$ at general points (λ^*, x^*) which satisfy (1.1), and it is not required that $G(\lambda, 0)$ should be identically zero.

We assume that a curve Γ of solutions, $(\lambda, x(\lambda))$ is being computed by a continuation method based on incrementing λ e.g. the Euler-Newton method of (1.5). The problem of implementing this efficiently is a very important topic in its own right, but it is not discussed here. We refer to Davis [9], Rheinboldt [25], and especially Abbott [1] and the references contained therein. Most continuation methods will encounter difficulties as a singular point (λ^*, x^*) on Γ is approached and special methods must be used to deal with them. In this chapter we shall assume that the following conditions are satisfied at (λ^*, x^*).

a) $\mathcal{N}\{G_x(\lambda^*, x^*)\}$ is 1-dimensional and spanned by ϕ^*, $\|\phi^*\| = 1$

b) $\mathcal{N}\{G_x(\lambda^*, x^*)'\}$ is 1-dimensional and spanned by ψ^*, $\|\psi^*\| = 1$ (3.1.1)

c) $\mathcal{R}\{G_x(\lambda^*, x^*)\}$ is closed

d) $G_\lambda(\lambda^*, x^*) \notin \mathcal{R}\{G_x(\lambda^*, x^*)\}$.

In the next two chapters we shall consider singular points which do not satisfy (3.1.1d).

In similar fashion to chapter 2, (3.1.1) allows us to infer the existence of $z^* \in X$ and $l^* \in X'$, satisfying (2.3.1). These define the projections P and Q of (2.1.4) and the decomposition of X into direct sums as in (2.1.5). We again denote the bounded inverse of $G_x(\lambda^*, x^*)$, as an operator from N to $\mathcal{R}\{G_x(\lambda^*, x^*)\}$, by M.

Now it is easy to carry out a Liapunov-Schmidt analysis to determine the behaviour of Γ near (λ^*, x^*). Writing (1.1) as

$$G_x(\lambda^*, x^*)\,\delta x + \delta\lambda\, G_\lambda(\lambda^*, x^*) + R(\delta\lambda, \delta x) = 0\,, \qquad (3.1.2)$$

where $\delta x = x - x^*$, $\delta\lambda = \lambda - \lambda^*$ and R is a continuously differentiable higher-order term satisfying

$$\|R(\delta\lambda, \delta x)\| = o(|\delta\lambda| + \|\delta x\|)\,, \qquad (3.1.3)$$

we proceed as in section 2.2 and seek solutions of the form $\delta x = \varepsilon\phi^* + w$, $w \in N$. Thus (3.1.2) is equivalent to the pair of equations

$$\begin{aligned} &\text{a)} \quad w + \delta\lambda\, MPG_\lambda(\lambda^*, x^*) + MPR(\delta\lambda, \varepsilon\phi^* + w) = 0 \\ &\text{b)} \quad \delta\lambda\, \psi^*(G_\lambda(\lambda^*, x^*)) + \psi^*(R(\delta\lambda, \varepsilon\phi^* + w)) = 0\,. \end{aligned} \qquad (3.1.4)$$

For sufficiently small $\delta\lambda$ and ε, (3.1.4a) defines, by the Implicit Function theorem, a unique small solution $w(\delta\lambda, \varepsilon)$ with

$$w(\delta\lambda, \varepsilon) = -\delta\lambda\, MPG_\lambda(\lambda^*, x^*) + o(|\delta\lambda| + |\varepsilon|)\,. \qquad (3.1.5)$$

This solution can be inserted into (3.1.4b) to give

$$a\,\delta\lambda + R_1(\delta\lambda, \varepsilon) = 0\,, \qquad (3.1.6)$$

where $a = \psi^*(G_\lambda(\lambda^*, x^*))$ is non-zero by (3.1.1d) and R_1 is a continuously differentiable function from R^2 to R satisfying

$$|R_1(\delta\lambda, \varepsilon)| = o(|\delta\lambda| + |\varepsilon|)\,. \qquad (3.1.7)$$

Thus, for sufficiently small ε, we can apply the Implicit Function theorem to (3.1.6) and obtain a unique small solution $\delta\lambda(\varepsilon)$, which is continuously differentiable with

$$\frac{d}{d\varepsilon}\delta\lambda(0) = 0\,. \qquad (3.1.8)$$

So we have a unique solution curve $\left(\lambda^* + \delta\lambda(\varepsilon), x^* + \varepsilon\phi^* + w(\delta\lambda(\varepsilon), \varepsilon)\right)$

of (1.1) passing through (λ^*, x^*) , but the tangent to the curve is normal to the λ-axis at this point (see Fig. 1.3) and straightforward use of λ as a continuation parameter is neither advisable nor, in general, successful. In the literature (λ^*, x^*) is usually called a turning point.

In the next section we briefly look at some of the techniques used to continue the solution curve through turning points and then, in section 3.3, at methods of actually computing the turning point itself. In sections 3.4 and 3.5 a new method for this problem is described, which can be thought of as a generalisation of the well-known application of Newton's method to the matrix eigen-value problem. Once a turning point has been determined, we show, in section 3.6, how the solution curve may be continued by methods analogous to those of chapter 2. Finally numerical results are presented in section 3.7.

3.2 Following Solution Paths through Turning Points

As stated above it is not possible to continue the solution curve through a turning point (λ^*, x^*) by incrementing λ, and then computing $x(\lambda)$. The obvious answer to this problem is to use a continuation method based on incrementing a different parameter, and there are two closely related methods of doing this.

3.2.1 Explicit Change of Parameter

This method was introduced almost simultaneously by Keller and Wolfe [20], Anselone and Moore [2], who used a parameter which occurred naturally in their problem, and Davis [9], who presented the idea abstractly. The basis of the method is to introduce a new real parameter ρ and to make λ a function of both x and ρ i.e.

$$H(\rho, x) = G(\lambda(\rho, x), x) \,. \tag{3.2.1}$$

Now if (λ°, x°) is a known solution near the turning point, and $\lambda(\rho^\circ, x^\circ) = \lambda^\circ$, we may attempt to continue the solution curve by solving

$$H(\rho, x) = 0 \tag{3.2.2}$$

by means of a continuation method based on incrementing ρ, and starting at (ρ°, x°). Then λ is recovered by the transformation $\lambda = \lambda(\rho, x(\rho))$. Of course for this to be successful we must have $H_x(\rho, x)$ non-singular near the turning point and $\lambda(\rho, x)$ is chosen with this in mind. Writing

$$H_x(\rho, x) = G_x(\lambda(\rho,x), x) + G_\lambda(\lambda(\rho,x), x)\,\frac{\partial}{\partial x}\lambda(\rho, x)\,, \tag{3.2.3}$$

we see that $\frac{\partial}{\partial x}\lambda(\rho,x)$ is the important function, and so attention may be restricted to linear transformations of the form

$$\lambda(\rho, x) = \rho + l(x)\,, \tag{3.2.4}$$

where l is a bounded linear functional on X. Of course (3.2.4) defines a linear homeomorphism of R x X onto itself by means of the mapping $(\rho, x) \to (\lambda(\rho, x), x)$ (see Davis) and hence it is simple to switch between (ρ, x) and (λ, x). If $\rho^* = \lambda^* - l(x^*)$ it also easily follows, using (3.1.1d) and (3.2.3), that $\mathcal{N}\{H_x(\rho^*, x^*)\} = \{0\}$ and $\mathcal{R}\{H_x(\rho^*, x^*)\} = X$ iff $\frac{\partial}{\partial x}\lambda(\rho^*, x^*)\phi^* \neq 0$ i.e.

$$l(\phi^*) \neq 0\,. \tag{3.2.5}$$

If $H_x(\rho^*, x^*)$ is non-singular, then $H_x(\rho, x)$ is non-singular in a neighbourhood of (ρ^*, x^*) by continuity, and this will include (ρ°, x°) if sufficiently close.

As can be seen the particular l used is not critical but a natural choice when $X = R^n$ is

$$l(x) = \underset{\sim}{e}_r^T \underset{\sim}{x}\,, \tag{3.2.6}$$

where $\underset{\sim}{e}_r$ is the r^{th} column of the identity matrix I_n and r maximises the component of ϕ^* i.e. $|\phi^*_r| = \max\{|\phi^*_i|\}$ $i=1,n$. If $G_x(\lambda, x)$ is a sparse matrix this will limit the additional elements needed to form the matrix $H_x(p, x)$, see (3.2.3), to a single column. Of course in practice we do not know ϕ^*, but the Liapunov-Schmidt analysis of section 3.1 shows that, near the turning point, the difference between points on Γ is primarily in the ϕ^*- direction. Thus we have a first approximation from which r can be estimated. Alternatively or additionally, the eigen-vector ϕ^0 corresponding to the smallest eigen-value of $G_x(\lambda^0, x^0)$ may be computed by inverse iteration and, by continuity, this will give r.

3.2.2 Implicit Change of Parameter

Various forms of this method have been used by Riks [26], Keller [17], Abbott [1] and Menzel and Schwetlich [23]. The idea is to regard λ and x as independent variables, and to introduce an extra scalar equation

$$b(\lambda, x) = 0 , \tag{3.2.7}$$

which determines the increment taken and which makes the solution of the augmented system ,

$$\begin{aligned} &a) \quad G(\lambda, x) = 0 \\ &b) \quad b(\lambda, x) = 0 , \end{aligned} \tag{3.2.8}$$

well-defined in the neighbourhood of the turning point. It is sufficient for b to be linear in λ and x and the usual continuation in λ is equivalent to the choice.

$$b(\lambda, x) = (\lambda - \lambda^0) - c , \tag{3.2.9}$$

and thus just solving $G(\lambda^0 + c, x) = 0$ for x . In order to pass through a turning point we must allow a more general choice of $b(\lambda, x)$, i.e

$$b(\lambda, x) = l(x - x^0) + \alpha(\lambda - \lambda^0) - c \tag{3.2.10}$$

where $l \in X'$ and α a real scalar.

It is easier to analyse (3.2.8) if we introduce the product space $Y = R \times X$ with some suitable $\|\ \|_Y$ i.e. $\|y\| = \max.\{|\lambda|, \|x\|\}$ where $y=(\lambda,x)$. Now (3.2.8), with l given by (3.2.10), defines a mapping from Y to itself consisting of

$$\begin{aligned} &\text{a)} \quad \tilde{G}(y) = 0 \\ &\text{b)} \quad \tilde{l}(y-y^0) - c = 0\,, \end{aligned} \qquad (3.2.11)$$

where $\tilde{G}(y) = G(\lambda,x)$ and $\tilde{l}$ is the element of Y' defined by l and α, which without loss of generality we assume to be of unit norm. If $y=(\lambda,x)$ satisfies $\tilde{G}(y)=0$ and is either a non-singular solution, i.e $G_x(\lambda,x)$ non-singular, or a turning point, it is easy to see that the linear equation

$$G_x(\lambda,x)u + \mu G_\lambda(\lambda,x) = 0 \qquad \mu\in R,\ u\in X\,, \qquad (3.2.12)$$

or equivalently $\tilde{G}_y(y)z = 0$ with $z\in(\mu,u)$, has a unique normalised non-trivial solution, which we denote by $z^* = (\mu^*,u^*)$. Also (3.2.11) linearised at y is non-singular **iff**

$$\tilde{l}^*(z^*) \neq 0\,. \qquad (3.2.13)$$

As $\tilde{l}$ and z^* are both of unit norm, the choice of $\tilde{l}$ to make (3.2.11) as well-posed as possible, in a certain sense, will be that for which

$$\tilde{l}(z^*) = 1\,. \qquad (3.2.14)$$

The existence of such $\tilde{l}$ is guaranteed by the Hahn-Banach theorem.

In the finite-dimensional case, a theorem relating the determinant of the $(n+1)\times(n+1)$ matrix, defined by (3.2.11) linearised, to $\tilde{l}(\underset{\sim}{z}^*)$ has been proved by Abbott. Abbott also gives methods for calculating $\underset{\sim}{z}^*$ and suggests that the choice

$$\tilde{l}(\underset{\sim}{y}) = \underset{\sim}{e}_r^T \underset{\sim}{y}\,, \qquad (3.2.15)$$

where $\underset{\sim}{e}_r$ is the r^{th} column of I_{n+1} and maximises the component of $\underset{\sim}{z}^*$, is more efficient as the $(n+1)\times(n+1)$ system can be immediately reduced to an $n\times n$ system. This is very similar to the explicit change of parameter, with $\lambda(p, \underset{\sim}{x}) = p + \underset{\sim}{e}_r^T \underset{\sim}{x}$, the difference being that, instead of the vector $G_\lambda(\lambda, \underset{\sim}{x})$ being added onto the r^{th} column of $G_x(\lambda, \underset{\sim}{x})$, it now replaces it.

3.3 Determination of Simple Turning Points

In this section we briefly describe two of the methods suggested for computing turning points themselves, in the finite-dimensional case, and then point out the significance of what we term "simple" turning points.

The method of Simpson [28] is representative of several techniques based on interpolation. $G_x(\lambda, \underset{\sim}{x})$ is assumed to be symmetric, so that 0 is a simple eigenvalue of $G_x(\lambda^*, \underset{\sim}{x}^*)$ and $(\lambda^*, \underset{\sim}{x}^*)$ is assumed to be a "limit point", i.e. in a neighbourhood of $(\lambda^*, \underset{\sim}{x}^*)$ the solution curve Γ only exists on one side of λ^* ($\lambda \geq \lambda^*$ or $\lambda \leq \lambda^*$) see Fig. 1.3a. (we shall assume that it only exists for $\lambda \leq \lambda^*$). Let $(\lambda_i, \underset{\sim}{x}_i)$ $i = 1,3$ be known points on Γ near $(\lambda^*, \underset{\sim}{x}^*)$, and $\mu(\lambda_i, \underset{\sim}{x}_i)$ the smallest eigen-value, in modulus, of $G_x(\lambda_i, \underset{\sim}{x}_i)$. A quadratic interpolating polynormial , $P(\mu)$, is passed through the points $(\lambda_i, \mu(\lambda_i, \underset{\sim}{x}_i))$ and used to approximate the inverse function $\lambda(\mu)$. The next approximation, λ_4, to the turning point is then given by

$$\lambda_4 = \alpha P(0) + (1-\alpha)\lambda_1 , \tag{3.3.1}$$

where the purpose of the scalar α, chosen in $(0,1)$, is to prevent $\lambda_4 > \lambda^*$. The procedure is then repeated with the three largest values of λ_i $i = 1,4$. The smallest eigen-value $\mu(\lambda_4, \underset{\sim}{x}(\lambda_4))$ can be calculated by a combination of Newton's method and the Rayleigh quotient method.

a) $G_x(\lambda_4, \underset{\sim}{x}^n)\delta\underset{\sim}{x}^n = -G(\lambda_4, \underset{\sim}{x}^n)$

b) $\mu^n = \langle \delta\underset{\sim}{x}^n, G_x(\lambda_4, \underset{\sim}{x}^n)\delta\underset{\sim}{x}^n \rangle / \langle \delta\underset{\sim}{x}^n, \delta\underset{\sim}{x}^n \rangle$ (3.3.2)

c) $\underset{\sim}{x}^{n+1} = \underset{\sim}{x}^n + \delta\underset{\sim}{x}^n$

with $\underset{\sim}{x}^{n+1} \to \underset{\sim}{x}(\lambda_4)$ and $\mu^n \to \mu(\lambda_4, \underset{\sim}{x}(\lambda_4))$ as $n \to \infty$. The convergence rate of the λ_i to λ^* is linear with ratio $(1-\alpha)$.

In Abbott's [1] method, which does not rely on symmetry, equation (1.1), which determines the solution curve, is augmented by an additional equation,

$$g(\lambda, \underset{\sim}{x}) = 0 \qquad g: R \times R^n \to R \quad , \tag{3.3.3}$$

which ensures that the turning point is the locally unique solution of the augmented system. The most obvious choice for $g(\lambda, \underset{\sim}{x})$ is $\det\left(G_x(\lambda, \underset{\sim}{x})\right)$, but Abbott gives several alternatives which will often be superior. The complete system

$$\begin{aligned} &a) \quad G(\lambda, \underset{\sim}{x}) = 0 \\ &b) \quad g(\lambda, \underset{\sim}{x}) = 0 \end{aligned} \tag{3.3.4}$$

is solved by an approximation to Newton's method which reflects the fact that $g(\lambda, \underset{\sim}{x})$ is expensive to evaluate and $g_x(\lambda, \underset{\sim}{x})$ not available. For each iteration, first a Newton step is taken with (3.3.4a),

$$G_x(\lambda^p, \underset{\sim}{x}^p)\,\delta\underset{\sim}{x}^p + \delta\lambda^p G_\lambda(\lambda^p, \underset{\sim}{x}^p) = -G(\lambda^p, \underset{\sim}{x}^p), \tag{3.3.5}$$

then this linear system of n equations in $(n+1)$ unknowns is solved for $\delta x_1, \ldots, \delta x^p_{r-1}, \delta x^p_{r+1}, \ldots, \delta x^p_n$ and $\delta\lambda^p$, in terms of δx^p_r, where r is chosen to make the resulting $n \times n$ matrix as well-conditioned as possible (see (3.2.15)). Now, after substitution, (3.3.4b) becomes a single non-linear scalar equation,

$$h(\delta x^p_r) = 0 \quad , \tag{3.3.6}$$

and a single step is taken with the approximate Newton's method

$$\delta x^p_r = -\left\{(h(\delta) - h(0))/\delta\right\}^{-1} h(0) \quad , \tag{3.3.7}$$

where $\delta \neq 0$ is some small scalar. With suitable choice of δ Abbott shows that quadratic convergence is attained.

We now regard X as a general Banach space again, and $G(\lambda, x)$

twice continuously differentiable, and make the following definition.

Definition 3.1

(λ^*, x^*) is a SIMPLE turning point of a solution curve Γ if, in addition to (3.1.1.),

$$G_{xx}(\lambda^*, x^*)\phi^*\phi^* \notin R\{G_x(\lambda^*, x^*)\} . \tag{3.3.8}$$

Both Simpson's and Abbott's results only apply to simple turning points and we point out their significance. Using (3.3.8), the Liapunov-Schmidt analysis of section 3.1 can be specialised and (3.1.6) replaced by

$$a\,\delta\lambda + b\,\varepsilon^2 + R_2(\delta\lambda, \varepsilon) = 0, \tag{3.3.9}$$

where $b = \psi^*(G_{xx}(\lambda^*,x^*)\phi^*\phi^*)/2$ and R_2 is twice continuously differentiable with

$$|R_2(\delta\lambda, \varepsilon)| = o(|\delta\lambda| + |\varepsilon|^2). \tag{3.3.10}$$

Thus, if we parametrise Γ by ε near (λ^*, x^*) to obtain $(\lambda(\varepsilon), x(\varepsilon))$, then

$$\frac{d^2}{d\varepsilon^2}\lambda(0) \neq 0 , \tag{3.3.11}$$

and we see that Γ must be of quadratic type, see Fig. 1.3a, with

$$\| x(\varepsilon) - x^* \| = O(|\lambda(\varepsilon) - \lambda^*|^{1/2}) . \tag{3.3.13}$$

However, more important that this is the following theorem, which shows that the singular point is isolated.

Theorem 3.2.

If (λ^*, x^*) is a simple turning point then, for $|\varepsilon| \neq 0$ sufficiently small, $G_x(\lambda(\varepsilon), x(\varepsilon))$ is non-singular and

$$\| G_x(\lambda(\varepsilon))x(\varepsilon))^{-1} \| = O(|\varepsilon|^{-1}) . \tag{3.3.13}$$

Proof

$$G_x(\lambda(\varepsilon), x(\varepsilon)) = G_x(\lambda^*, x^*) + \varepsilon G_{xx}(\lambda^*, x^*)\phi^* + \varepsilon L(\varepsilon) \tag{3.3.14}$$

where $\|L(\varepsilon)\| \to 0$ as $\varepsilon \to 0$. Lemma 2.1 shows that $G_x(\lambda^*, x^*) + \varepsilon G_{xx}(\lambda^*, x^*)\phi^*$ satisfies the theorem and the Banach perturbation theorem takes care of the extra term $\varepsilon L(\varepsilon)$ ■

This theorem may be compared with Theorem 2.2 for bifurcation from the trivial solution.

It is easy to see what may happen if (λ^*, x^*) is not a simple turning point. The solution curve Γ is tangential to the surface on which $G_x(\lambda, x)$ is singular, and may even be contained in it. An example of this is the linear case (1.2), where $(\lambda^*, \alpha\phi^*)$ is a turning point for all non-zero α.

3.4 A New Method for Simple Turning Points

The basic idea of the method is to solve the three equations

$$\begin{aligned} &a) \quad G(\lambda, x) = 0 \\ &b) \quad G_x(\lambda, x)\phi = 0 \\ &c) \quad l(\phi) - 1 = 0 \end{aligned} \tag{3.4.1}$$

for $\lambda \in R$ and $x, \phi \in X$ by the Newton-Kantorovich method. l is a scaling bounded linear functional chosen so that

$$l(\phi^*) \simeq 1 . \tag{3.4.2}$$

Of course in practice we do not know ϕ^* but, so long as $l(\phi^*) \neq 0$, the choice of l will determine which element of $\mathcal{N}\{G_x(\lambda^*, x^*)\}$ the method converges to. We denote this element by $\bar{\phi}^*$.

In order to analyse (3.4.1) we define $y = (\lambda, x, \phi)$ and re-write the equation as

$$F(y) = 0 \tag{3.4.3}$$

where $F : R \times X \times X \to X \times X \times R$. We formally identify these two spaces as Y, which becomes a Banach space under the norm

$$\|y\|_Y = \max.\{|\lambda|, \|x\|, \|\phi\|\} . \tag{3.4.4}$$

This leads to the following important Lemma.

Lemma 3.3

If (λ^*, x^*) is a simple turning point and $y^* = (\lambda^*, x^*, \bar{\phi}^*)$, then $F_y(y^*)^{-1}$ exists as a bounded linear operator on Y .

Proof

We use the notation carried over from chapter 2 and referred to after (3.1.1).

Now, if $F_y(y^*)y_1 = y_2$, with $y_1 = (\lambda, x, z) \in R \times X \times X$ and $y_2 = (u, v, \mu) \in X \times X \times R$, we have

$$\begin{aligned} &a) \quad G_x(\lambda^*, x^*)x + G_\lambda(\lambda^*, x^*)\lambda = u \\ &b) \quad G_x(\lambda^*, x^*)z + G_{xx}(\lambda^*, x^*)\bar{\phi}^* x + \lambda G_{x\lambda}(\lambda^*, x^*)\bar{\phi}^* = v \\ &c) \quad l(z) = \mu . \end{aligned} \tag{3.4.5}$$

Using simple algebra these equations can be reformulated to

$$\begin{aligned} &a) \ (i) \quad \lambda = \psi^*(u)/\psi^*(G_\lambda(\lambda^*, x^*)) \\ &\quad\ (ii) \quad Qx = M\{Pu - \lambda PG_\lambda(\lambda^*, x^*)\} \\ &b) \ (i) \quad l^*(x) = \{\psi^*(v) - \psi^*(G_{xx}(\lambda^*, x^*)\bar{\phi}^* Qx) - \lambda\psi^*(G_{x\lambda}(\lambda^*, x^*)\bar{\phi}^*)\} \\ &\qquad\qquad /\psi^*(G_{xx}(\lambda^*, x^*)\bar{\phi}^*\phi^*) \\ &\quad\ (ii) \quad Qz = M\{Pv - \lambda PG_{x\lambda}(\lambda^*, x^*)\bar{\phi}^* - PG_{xx}(\lambda^*, x^*)\bar{\phi}^* x\} \\ &c) \quad l^*(z) = \{\mu - l(Qz)\}/l(\phi^*) . \end{aligned} \tag{3.4.6}$$

As $\psi^*(G_\lambda(\lambda^*, x^*))$, $\psi^*(G_{xx}(\lambda^*, x^*)\bar{\phi}^*\phi^*)$ and $l(\phi^*)$ are all non-zero by assumption, it is clear, by construction, that $F_y(y^*)$ is 1 - 1 and $\mathcal{R}\{F_y(y^*)\} = Y$, and hence the open-mapping theorem shows that $F_y(y^*)^{-1}$ is a bounded linear operator on Y ∎

Now that we know $F(y^*) = 0$, and $F_y(y^*)^{-1}$ exists, we can apply Theorem 2.3 to prove the convergence of Newton's method.

Theorem 3.4

If (λ^*, x^*) is a simple turning point, and the second derivatives G_{xx} and $G_{x\lambda}$ satisfy a uniform Lipschitz condition for $|\lambda - \lambda^*| \leq \tau_1$ and $\|x - x^*\| \leq \tau_1$, then there exists $0 < \tau \leq \tau_1$, such that if $\|y^0 - y^*\|_y \leq \tau$ the Newton iteration for (3.4.3),

$$y^{n+1} = y^n - F_y(y^n)^{-1} F(y^n) \qquad n = 0,1,2, \tag{3.4.7}$$

converges quadratically to the turning point y^*.

Proof

Let $y^0 = (\lambda^0, x^0, \phi^0)$ be the initial approximation. Expanding about y^* gives

$$\begin{aligned}
&\text{a)}\ \|G(\lambda^0, x^0)\| \leq \|G_x(\lambda^*, x^*)\| \|x^0 - x^*\| + \|G_\lambda(\lambda^*, x^*)\| |\lambda^0 - \lambda^*| + 2K_1\tau^2 \\
&\text{b)}\ \|G_x(\lambda^0, x^0)\phi^0\| \leq \|G_x(\lambda^*, x^*)\| \|\phi^0 - \bar{\phi}^*\| + 2K_1\tau(a+\tau) \\
&\text{c)}\ |l(\phi^0) - 1| \leq \|l\|\tau,
\end{aligned} \tag{3.4.8}$$

where $a = \|\bar{\phi}^*\|$ and K_1 is a uniform bound on the second derivatives of G for $|\lambda - \lambda^*| \leq \tau_1$ and $\|x - x^*\| \leq \tau_1$. Thus there exists $d_1 > 0$ such that

$$\|F(y^0)\|_Y \leq d_1 \tau. \tag{3.4.9}$$

From Lemma 3.3

$$\|F_y(y^*)^{-1}\| \leq d_2 \tag{3.4.10}$$

and, if K_2 is the uniform Lipschitz constant for G_{xx} and $G_{x\lambda}$, it easily follows that

$$\|F_y(y^0) - F_y(y^*)\|_Y \leq 4(K_1 + K_2 a)\|y^0 - y^*\|_Y. \tag{3.4.11}$$

Thus if $\delta_\tau = 1 - 4(K_1 + K_2)d_2\tau > 0$ we can use the Banach perturbation theorem to prove that $F_y(y^0)^{-1}$ has a bounded inverse, and

$$\|F_y(y^0)^{-1}\|_Y \leq d_2 \delta_\tau^{-1}. \tag{3.4.12}$$

Using the notation of Theorem 2.3, $\eta = d_1 d_2 \delta_\tau^{-1} \tau$, and so to obtain a Lipschitz constant for $F_y(y)$ with $\|y - y^0\| < 2\eta$, we must restrict τ

so that

$$\tau(1+2d_1 d_2 \delta_\tau^{-1}) < \tau_1 . \qquad (3.4.13)$$

With τ so restricted it follows that

$$\| F_y(z_1) - F_y(z_2) \|_Y \leqslant 4(K_1+K_2 a) \| z_1 - z_2 \|_Y \qquad (3.4.14)$$

for $\| z_i - y_0 \|_Y < 2\eta \quad i=1,2$.

Finally, (3.4.9), (3.4.12) and (3.4.14) imply that

$$h \leqslant 4 d_1 (d_2/\delta_\tau)^2 (K_1+K_2 a)\tau , \qquad (3.4.15)$$

and thus Theorem 2.3 can be applied by restricting τ so that

a) $\delta_\tau > 0$

b) $\tau < \min\{\tau_1 \delta_\tau (\delta_\tau + 2d_1 d_2)^{-1}, \tau_1/2\}$ (3.4.16)

c) $h < \frac{1}{2}$ ∎

If a uniform Lipschitz constant does not hold for G_{xx} and $G_{x\lambda}$ then a similar theorem can still be proved but quadratic convergence cannot be guaranteed.

3.5 Implementation of the Method

In most situations $G(\lambda, x)$ will be a system of n non-linear equations derived from the discretization of the model of a physical problem; thus in this section we assume $X = R^n$ and write $\underset{\sim}{x}, \underset{\sim}{\phi}$ etc. An efficient numerical procedure for solving the linear systems associated with the Newton iteration (3.4.7) is developed but first we will find it helpful to look at the linear eigen-value problem.

In [31] Wilkinson comments on the equivalence of inverse iteration and Newton's method for find the eigen-value μ and associated eigen-vector $\underset{\sim}{u}$ of a matrix A. With starting value $\mu^0, \underset{\sim}{u}^0$ and first component normalised, the iterations are

Inverse Iteration:-

a) $(A-\mu^0 I)(\underset{\sim}{u}^0 + \delta\underset{\sim}{u}) = \delta\mu\, \underset{\sim}{u}^0$

b) $\underset{\sim}{e}_1^T \delta \underset{\sim}{u} = 0$ (3.5.1)

Newton's Method:-

$$\text{a)} \quad (A-\mu^\circ I)\delta \underset{\sim}{u} - \delta\mu \, \underset{\sim}{u}^\circ = -(A-\mu^\circ I)\underset{\sim}{u}^\circ$$
$$\text{b)} \quad \underset{\sim}{e}_1^T \delta \underset{\sim}{u} = 0 . \qquad (3.5.2)$$

As can easily be seen the two iterations are just reformulations of one another. We can write the Newton method as an $n \times n$ problem in the following way by taking advantage of the fact that $\delta u_1 = 0$.

$$A^* \delta \underset{\sim}{s} = -(A-\mu^\circ I)\underset{\sim}{u}^\circ \qquad (3.5.3)$$

where $\delta \underset{\sim}{s}^T = (\delta\mu, \delta u_2, \ldots, \delta u_n)$ and

$$A^* = \left(-\underset{\sim}{u}^\circ \,\vdots\, A-\mu^\circ I\right) \qquad (3.5.4)$$

i.e. A with its first column replaced by $-\underset{\sim}{u}^\circ$.

This method can easily be extended to generalized eigen-value problems where the matrix A depends on μ in a non-linear manner,

$$A(\mu)\underset{\sim}{u} = 0 . \qquad (3.5.5)$$

The Newton iteration

$$\text{a)} \quad A(\mu^\circ)\delta \underset{\sim}{u} + \delta\mu \frac{dA}{d\mu}(\mu^\circ)\underset{\sim}{u}^\circ = -A(\mu^\circ)\underset{\sim}{u}^\circ$$
$$\text{b)} \quad \underset{\sim}{e}_1^T \delta \underset{\sim}{u} = 0 \qquad (3.5.6)$$

can be written in the form

$$A^* \delta \underset{\sim}{s} = -A(\mu^\circ)\underset{\sim}{u}^\circ , \qquad (3.5.7)$$

where

$$A^* = \left(\frac{dA}{d\mu}(\mu^\circ)\underset{\sim}{u}^\circ \,\vdots\, A(\mu^\circ)\right) . \qquad (3.5.8)$$

In this section we will extend these ideas to our non-linear eigen-value problems.

Using a continuation method we obtain an approximation $(\lambda^\circ, \underset{\sim}{x}^\circ)$ to the turning point. For λ close to λ^* the change in $\underset{\sim}{x}(\lambda)$ is primarily in the $\underset{\sim}{\phi}^*$-direction, as can be seen from the Liapunov-Schmidt analysis, and this gives a first approximation $\underset{\sim}{\phi}^\circ$. Alternatively, if $G_x(\lambda, \underset{\sim}{x})$ is symmetric and the continuation method is based on Newton, the difference in successive iterates can be used, as in Simpson's [28] method. Either of these approximations can be refined using inverse iteration with $G_x(\lambda^0, \underset{\sim}{x}^0)$. Finally, therefore, an initial $\underset{\sim}{\phi}^0$ is obtained, which is assumed to be normalised so that its largest (in modulus) component ϕ_r^0 is unity. Now we define

$$l(\underset{\sim}{x}) = \underset{\sim}{e}_r^T \underset{\sim}{x}\,, \tag{3.5.9}$$

for l in (3.4.1c), and by construction $l(\underset{\sim}{\phi}^0) = 1$. Without loss of generality we shall assume forthwith that $r = 1$.

$\underset{\sim}{y}^0 = (\lambda^0, \underset{\sim}{x}^0, \phi^0)$ is the starting value for (3.4.7)

and, if the conditions of Theorem 3.4 are satisfied, the iteration

$$\begin{aligned} F_y(\underset{\sim}{y}^n)\,\delta\underset{\sim}{y}^n &= -F(\underset{\sim}{y}^n) \\ \underset{\sim}{y}^{n+1} &= \underset{\sim}{y}^n + \delta\underset{\sim}{y}^n\,, \end{aligned} \tag{3.5.10}$$

where $\delta\underset{\sim}{y}^n = (\delta\lambda, \delta\underset{\sim}{x}^n, \delta\underset{\sim}{\phi}^n)$, will converge to $\underset{\sim}{y}^* = (\lambda^*, \underset{\sim}{x}^*, \bar{\phi}^*)$. If we suppress the superscripts on $\delta\underset{\sim}{y}$, (3.5.10) can be written

$$\begin{aligned} &\text{a)} \quad A\,\delta\underset{\sim}{x} + \delta\lambda\,\underset{\sim}{d}_1 = \underset{\sim}{c}_1 \\ &\text{b)} \quad A\,\delta\underset{\sim}{\phi} + \delta\lambda\,\underset{\sim}{d}_2 + B\,\delta\underset{\sim}{x} = \underset{\sim}{c}_2 \\ &\text{c)} \quad \delta\phi_1 = 0\,, \end{aligned} \tag{3.5.11}$$

where

$$\begin{aligned} A &= G_x(\lambda^n, \underset{\sim}{x}^n) & B &= G_{xx}(\lambda^n, \underset{\sim}{x}^n)\underset{\sim}{\phi}^n \\ \underset{\sim}{d}_1 &= G_\lambda(\lambda^n, \underset{\sim}{x}^n) & \underset{\sim}{d}_2 &= G_{\lambda x}(\lambda^n, \underset{\sim}{x}^n)\underset{\sim}{\phi}^n \\ \underset{\sim}{c}_1 &= -G(\lambda^n, \underset{\sim}{x}^n) & \underset{\sim}{c}_2 &= -G_x(\lambda^n, \underset{\sim}{x}^n)\underset{\sim}{\phi}^n \end{aligned}$$

and $\delta\phi_1$ is the first component of $\delta\underset{\sim}{\phi}$. Now we let

$$\begin{aligned} &\text{a)} \quad \delta\underset{\sim}{s}^T = (\delta\lambda, \delta x_2, \ldots, \delta x_n) \\ &\text{b)} \quad \delta\underset{\sim}{t}^T = (\delta\lambda, \delta\phi_2, \ldots, \delta\phi_n) \end{aligned} \tag{3.5.12}$$

and

$$A^* = \left(\underset{\sim}{d}_1 \,\vdots\, A\right) \tag{3.5.13}$$

i.e. $G_x(\lambda^n, \underset{\sim}{x}^n)$ with first column replaced by $G_\lambda(\lambda^n, \underset{\sim}{x}_n)$.

Thus (3.5.11) can be re-written

$$\begin{aligned} &\text{a)} \quad A^*\delta\underset{\sim}{s} = \underset{\sim}{c}_1 - \delta x_1\,\underset{\sim}{a}_1 \\ &\text{b)} \quad A^*\delta\underset{\sim}{t} = \underset{\sim}{c}_2 - B\,\delta\underset{\sim}{x} + \delta\lambda(\underset{\sim}{d}_1 - \underset{\sim}{d}_2) \end{aligned} \tag{3.5.14}$$

where $\underset{\sim}{a}_1$ is the first column of A.

We can solve (3.5.14a) for $\delta\underset{\sim}{s}$ in terms of δx_1

i.e.

$$\delta\underset{\sim}{s} = \underset{\sim}{\alpha} + \delta x_1 \underset{\sim}{\beta} \ , \qquad (3.5.15)$$

or

$$\begin{aligned} &\text{a)} \quad \delta\lambda = \alpha_1 + \beta_1 \delta x_1 \\ &\text{b)} \quad \delta\underset{\sim}{x}^T = (\delta x_1, \alpha_2 + \beta_2 \delta x_1, \ldots, \alpha_n + \beta_n \delta x_1) \ . \end{aligned} \qquad (3.5.16)$$

Substituting (3.5.16) into (3.5.14b) gives

$$A^* \delta\underset{\sim}{t} = \underset{\sim}{b}_1 + \delta x_1 \underset{\sim}{b}_2 \ , \qquad (3.5.17)$$

and so we can solve for $\delta\underset{\sim}{t}$ in terms of δx_1 i.e.

$$\delta\underset{\sim}{t} = \underset{\sim}{\chi} + \delta x_1 \underset{\sim}{\eta} \ , \qquad (3.5.18)$$

or

$$\begin{aligned} &\text{a)} \quad \delta\lambda = \gamma_1 + \eta_1 \delta x_1 \\ &\text{b)} \quad \delta\underset{\sim}{\phi}^T = (0, \gamma_2 + \eta_2 \delta x_1, \ldots, \gamma_n + \eta_n \delta x_1) \ . \end{aligned} \qquad (3.5.19)$$

Finally the two simultaneous equations (3.5.16a) and (3.5.19a) can be solved for $\delta\lambda$ and δx_1 and thus $\delta\underset{\sim}{x}$ and $\delta\underset{\sim}{\phi}$ obtained from (3.5.16b) and (3.5.19b) respectively. This completes one step of the Newton iteration (3.5.10) and we repeat until sufficient accuracy is attained.

We conclude this section with a number of remarks.

REMARK 1: The calculation of each new iterate requires the solution of four $n \times n$ linear systems (3.5.14a) and (3.5.17), each with the same coefficient matrix A^*. Thus only one LU decomposition need be performed at each stage and for most problems this compares favourably with Abbott's method.

REMARK 2: Of course whether this new method is suitable for a particular problem will depend to a great extent on how difficult it is to compute

$G_{xx}(\lambda,x)$ and $G_{x\lambda}(\lambda,x)$. For mildly non-linear problems of the form considered by Simpson, or for Hammerstein integral equations, these calculations can be quite easy; for large-scale finite element applications they may be impossible. However it should always be borne in mind that approximations involving only first derivatives are available ie.

$$\begin{aligned} &\text{a)} \quad G_{xx}(\lambda,x)\,z_1 z_2 = \lim_{\varepsilon\to 0}\{\varepsilon^{-1} G_x(\lambda, x+\varepsilon z_2) z_1\} \\ &\text{b)} \quad G_{\lambda x}(\lambda,x)\,z = \lim_{\varepsilon\to 0}\{\varepsilon^{-1} G_\lambda(\lambda, x+\varepsilon z)\}\,. \end{aligned} \tag{3.5.20}$$

REMARK 3: A difficulty with turning point problems is that the original matrix A is modified to A^* by a rank-1 change involving the alteration of a particular column. In the case of a full matrix, derived say from an integral equation, this is no problem, and similarly if A is a general sparse matrix. However, if A has diagonal structure, which we are utilising to solve the linear systems, this will be destroyed by the modification. Thus, for symmetric problems, it may well be more efficient to remove both the r^{th} row and r^{th} column of A and retain the diagonal structure, although this means that six $(n-1)\times(n-1)$ linear systems must be solved at each iteration.

REMARK 4: This method will also cope quite well with non-simple turning points. As A^* is non-singular for a general turning point, because of (3.1.1d), the near singularity of F_y will only appear in the 2 x 2 system for $\delta\lambda$ and δx_1 and therefore can be catered for.

REMARK 5: The modified Newton method, Theorem 2.5, can be applied to this problem and may be preferable in certain cases.

3.6 Moving away from Simple Turning Points

In this section we assume that a simple turning point (λ^*, x^*) on the solution curve Γ has already been determined and now we wish to continue Γ away from it.

Using the Liapunov-Schmidt analysis of section 3.3, and specifically (3.3.9), we see that the two half-branches of Γ emanating from (λ^*, x^*) may be parametrised individually by $\delta\lambda$ i.e.

$$x_i(\delta\lambda) = x^* + \varepsilon_i(\delta\lambda)\phi^* + w(\delta\lambda, \varepsilon_i(\delta\lambda)) \qquad i=1,2, \qquad (3.6.1)$$

where $\varepsilon_i(\delta\lambda) = (-1)^i(-a\delta\lambda/b)^{1/2} + o(|\delta\lambda|^{1/2})$, and $\delta\lambda$ is restricted to the sign of $(-a/b)$. Using the lowest order terms of (3.6.1) as a starting approximation, Newton's method can be applied as in section 2.6.

Theorem 3.5

For fixed $\delta\lambda$ sufficiently small, and of the same sign as $(-a/b)$, the Newton iteration

$$x^{n+1}(\delta\lambda) = x^n(\delta\lambda) - G_x(\lambda, x^n(\delta\lambda))^{-1} G(\lambda, x^n(\delta\lambda)), \qquad (3.6.2)$$

with starting approximation

$$x^0(\delta\lambda) = x^* \pm (-a\,\delta\lambda/b)^{1/2}\phi^*, \qquad (3.6.3)$$

converges to the solution curve Γ.

Proof

We verify the conditions of Corollary 2.4.

$$G_x(\lambda, x^0(\delta\lambda)) = G_x(\lambda^*, x^*) \pm (-a\delta\lambda/b)^{1/2} G_{xx}(\lambda^*, x^*)\phi^* + L(\delta\lambda) \qquad (3.6.4)$$

where $\|L(\delta\lambda)\| = o(|\delta\lambda|^{1/2})$. Thus we can use Lemma 2.1 and the Banach perturbation theorem to show that $G_x(\lambda, x^0(\delta\lambda))$ has a bounded inverse, for sufficiently small non-zero $\delta\lambda$, and

$$\|G_x(\lambda, x^\circ(\delta\lambda))^{-1}\| = O(|\delta\lambda|^{-1/2}), \qquad (3.6.5)$$

but for $x \in \mathcal{R}\{G_x(\lambda^*, x^*)\}$

$$\|G_x(\lambda, x^\circ(\delta\lambda))^{-1}x\| \leqslant K\|x\|. \qquad (3.6.6)$$

$$G(\lambda, x^\circ(\delta\lambda)) = \delta\lambda\{G_\lambda(\lambda^*, x^*) - \tfrac{1}{2}(a/b)G_{xx}(\lambda^*, x^*)\phi^*\phi^*\} + o(|\delta\lambda|) \qquad (3.6.7)$$

and so, by definition of a and b

$$\text{(i)} \quad \|G(\lambda, x^\circ(\delta\lambda))\| = O(|\delta\lambda|)$$

$$\text{but (ii)} \quad |\psi^*(G(\lambda, x^\circ(\delta\lambda)))| = o(|\delta\lambda|). \qquad (3.6.8)$$

Combining (3.6.5), (3.6.6) and (3.6.8) gives

$$\|G_x(\lambda, x^\circ(\delta\lambda))^{-1}G(\lambda, x^\circ(\delta\lambda))\| \leqslant \eta(\delta\lambda) = K_1(\delta\lambda)|\delta\lambda|^{1/2} \qquad (3.6.9)$$

where $K_1(\delta\lambda) \to 0$ as $\delta\lambda \to 0$.

If z_i $i=1,2$ satisfy $\|z_i - x^\circ(\delta\lambda)\| < 2\eta(\delta\lambda)$ then, using

$$G_x(\lambda, z_1) - G_x(\lambda, z_2) = \int_0^1 G_{xx}(\lambda, z_2 + t(z_1 - z_2))(z_1 - z_2)\,dt,$$

we have

$$\|G_x(\lambda, x^\circ(\delta\lambda))^{-1}\{G_x(\lambda, z_1) - G_x(\lambda, z_2)\}\| \leqslant K_2(\delta\lambda)|\delta\lambda|^{-1/2}\|z_1 - z_2\|, \qquad (3.6.10)$$

where $K_2(\delta\lambda) = O(1)$.

Because $K_1(\delta\lambda)K_2(\delta\lambda) \to 0$ as $\delta\lambda \to 0$ the conditions of Corollary 2.4 are satisfied and the theorem proved ∎

<u>Corollary 3.6</u>

If $G(\lambda, x)$ is three times continuously differentiable with respect to x the convergence rate is given by

$$\|x^n(\delta\lambda) - x^*(\delta\lambda)\| \leqslant (K_3|\delta\lambda|^{1/2})^{2^n - 1} K_4|\delta\lambda|/2^{n-1}. \qquad (3.6.11)$$

<u>Proof</u>

$\psi^*(G(\lambda, x^\circ(\delta\lambda))) = O(|\delta\lambda|^{3/2})$ and the result follows from the theorem ∎

3.7 Numerical Results

The method described in section 3.4 has been tested on several examples and two of these are given below. Conforming with earlier notation, we write $\underset{\sim}{y} = (\lambda, \underset{\sim}{x}, \underset{\sim}{\phi})$.

Example 1

Consider the mildly non-linear p.d.e.

$$\begin{aligned} &\text{a)} \quad -\Delta x(s) = \lambda e^{x(s)} && s \in D \\ &\text{b)} \quad x(s) = 0 && s \in \partial D \end{aligned} \qquad (3.7.1)$$

where D is the unit square $0 < s_1, s_2 < 1$, $s = (s_1, s_2)$. This problem is discussed in Simpson [28]. Let h be a uniform mesh steplength, $\Box_h$ the nine-point box form of the discrete Laplacian, and Δ_h the five-point form; then a discretization of (3.7.1) is

$$\Box_h \underset{\sim}{x} + \lambda\left\{ e^{\underset{\sim}{x}} + \frac{h}{12} \Delta_h e^{\underset{\sim}{x}} \right\} = 0 \qquad (3.7.2)$$

where $\underset{\sim}{x}$ is the mesh function.

With $h = 1/16$, Euler-Newton continuation was used to find the solution of (3.7.2) until $\lambda = 6.5$ and then the algorithm of section 3.4 was applied. The results, which differ slightly from those of Simpson but agree with those of Abbott [1], are given in Table 1. In Table 2 results are given for moving away from the turning point, as in section 3.6.

Table 1

λ	$\underset{\sim}{x}(\frac{1}{2},\frac{1}{2})$	$\lvert\delta\lambda\rvert = \lVert\delta y\rVert_\infty$	$\lVert\delta x\rVert_\infty$	$\lVert\delta\phi\rVert_\infty$
6.5	1.0042912488			
		0.69	0.39	0.42 E-2
7.1865083054	1.3967847490			
		0.37	0.13 E-1	0.63 E-2
6.8119819703	1.3922436541			
		0.39 E-2	0.59 E-3	0.17 E-3
6.8080884323	1.3916565122			
		0.19 E-5	0.20 E-6	0.80 E-7
6.8080865747	1.3916567083			
		0.12 E-12	0.18 E-13	0.88 E-14
6.8080865747	1.3916567083			

The following two values were computed at the turning point $(\lambda^*, \underset{\sim}{x}^*, \underset{\sim}{\phi}^*)$

$$
\begin{aligned}
&\text{a)} \quad \underset{\sim}{\phi}^{*T} G_\lambda(\lambda^*, \underset{\sim}{x}^*) = 0.38 \text{ E-1} \\
&\text{b)} \quad \underset{\sim}{\phi}^{*T} G_{xx}(\lambda^*, x^*)\phi^*\phi^* = 0.24 \text{ E-2} .
\end{aligned}
\tag{3.7.3}
$$

Table 2

	$\delta\lambda = -0.01$		$\delta\lambda = -0.05$	
r	$x(\frac{1}{2},\frac{1}{2})$	$\Vert\delta x\Vert_\infty$	$x(\frac{1}{2},\frac{1}{2})$	$\Vert\delta x\Vert_\infty$
0	1.446086337		1.513365058	
		0.30 E-1		0.73 E-1
1	1.475667702		1.586800388	
		0.53 E-2		0.15 E-1
2	1.470338138		1.572278116	
		0.19 E-3		0.67 E-3
3.	1.470145878		1.571604665	
		0.25 E-6		0.14 E-5
4	1.470145628		1.571603226	
		0.39 E-12		0.66 E-11
5	1.470145628		1.571603226	

Example 2

We consider again the two-point boundary value problem of section 2.8.

$$\text{a)}\quad x'' - \lambda(x + x^2 + x^3) = 0$$
$$\text{b)}\quad x(0) = x(1) = 0. \tag{3.7.4}$$

As noted in Example 3 there, we have a turning point close to the trivial solution and near the second eigen-value of the linearisation of (3.7.4). Starting from the approximation given by Table 3 of section 2.8, we compute this turning point below and give the results for moving beyond it in Table 4.

Table 3

λ	$\underset{\sim}{x}(1/4)$	$\lvert\delta\lambda\rvert$	$\lVert\delta \underset{\sim}{x}\rVert_\infty$	$\lVert\delta \underset{\sim}{\phi}\rVert_\infty$	$\lVert\delta \underset{\sim}{y}\rVert_\infty$
-39.4473100956	-0.1551600633				
		0.23 E-1	0.64 E-1	0.15	0.15
-39.4699287003	-0.2178889057				
		0.41 E-2	0.16 E-1	0.38 E-1	0.38 E-1
-39.4658396278	-0.2022088684				
		0.84 E-2	0.37 E-2	0.55 E-2	0.84 E-2
-39.4574530167	-0.2057502314				
		0.17 E-3	0.29 E-3	0.52 E-3	0.52 E-3
-39.4576245265	-0.2054685601				
		0.51 E-6	0.68 E-6	0.14 E-5	0.14 E-5
-39.4576240121	-0.2054678877				
		0.95 E-11	0.73 E-12	0.57 E-12	0.95 E-11
-39.4576240121	-0.2054678877				

The values of $\underset{\sim}{\phi}^{*T} G_\lambda(\lambda^*, \underset{\sim}{x}^*)$ and $\underset{\sim}{\phi}^{*T} G_{xx}(\lambda^*, \underset{\sim}{x}^*)\underset{\sim}{\phi}^*\underset{\sim}{\phi}^*$ were respectively 0.52 E-3 and -0.3 E-3.

Table 4

r	$\underset{\sim}{x}(1/4)$	$\lVert\delta \underset{\sim}{x}\rVert_\infty$	$\underset{\sim}{x}(1/4)$	$\lVert\delta \underset{\sim}{x}\rVert_\infty$
0	-0.2368245161		-0.2755834403	
		0.10 E-1		0.15 E-1
1	-0.2468293257		-0.2898986123	
		0.33 E-3		0.89 E-3
2	-0.2465049157		-0.2907275923	
		0.65 E-5		0.15 E-5
3	-0.2464985882		-0.2907261167	
		0.87 E-9		0.64 E-10
4	-0.2464985874		-0.2907261167	

CHAPTER 4

BIFURCATION FROM NON-TRIVIAL SOLUTIONS

4.1 Introduction

In this chapter we assume that the problem is the same as in chapter 3, i.e. we are computing a solution curve Γ of (1.1) by means of a continuation method based on incrementing λ but now a singular point (λ^*, x^*) is approached. However the conditions imposed on G at (λ^*, x^*) in this chapter are

a) $\mathcal{N}\{G_x(\lambda^*, x^*)\}$ is 1-dimensional and spanned by $\phi^*, \|\phi^*\| = 1$.

b) $\mathcal{N}\{G_x(\lambda^*, x^*)'\}$ is 1-dimensional and spanned by $\psi^*, \|\psi^*\| = 1$. (4.1.1)

c) $\mathcal{R}\{G_x(\lambda^*, x^*)\}$ is closed.

d) $G_\lambda(\lambda^*, x^*) \in \mathcal{R}\{G_x(\lambda^*, x^*)\}$.

It can be seen that these conditions are precisely those of (3.1.1) except that (4.1.1d) is the converse of (3.1.1d). This is generally regarded as the distinction between turning points and bifurcation points. As in chapter 3 we use the notation introduced after (2.1.2).

We now investigate the nature of the solution-set of (1.1) near (λ^*, x^*), under conditions (4.1.1), by means of the Liapunov-Schmidt technique. Equations (3.1.2) to (3.1.5) can be derived exactly as in chapter 3 but now, because of (4.1.1d), a = 0 in (3.1.6) and we need more information to obtain meaningful results. Thus we assume that G is twice continuously differentiable and (3.1.5) becomes

$$w(\delta\lambda, \varepsilon) = \delta\lambda\, w^* + O((|\delta\lambda| + |\varepsilon|)^2). \qquad (4.1.2)$$

(3.1.4b) can also be expanded to give

$$\begin{aligned} &\tfrac{1}{2}\delta\lambda^2 \psi^*(G_{\lambda\lambda}(\lambda^*, x^*)) + \delta\lambda\, \psi^*(G_{\lambda x}(\lambda^*, x^*)\delta x) \\ &\quad + \tfrac{1}{2}\psi^*(G_{xx}(\lambda^*, x^*)\delta x \delta x) + R_2(\delta\lambda, \delta x) = 0 , \end{aligned} \qquad (4.1.3)$$

where R_2 is twice continuously differentiable and

$$R_2(\delta\lambda,\delta x) = o\left((|\delta\lambda| + \|\delta x\|)^2 \right). \tag{4.1.4}$$

By analogy with the approach in chapter 2 we replace δx by $\varepsilon\phi^* + w(\delta\lambda,\varepsilon)$ in (4.1.3) and, using (4.1.2), this simplifies to

$$A\,\delta\lambda^2 + B\,\delta\lambda\,\varepsilon + C\varepsilon^2 + R_3(\delta\lambda,\varepsilon) = 0 \ , \tag{4.1.5}$$

where

a) $A = \frac{1}{2}\psi^*(G_{\lambda\lambda}(\lambda^*,x^*) + 2G_{\lambda x}(\lambda^*,x^*)w^* + G_{xx}(\lambda^*,x^*)w^*w^*)$

b) $B = \psi^*(G_{\lambda x}(\lambda^*,x^*)\phi^* + G_{xx}(\lambda^*,x^*)\phi^* w^*)$

c) $C = \frac{1}{2}\psi^*(G_{xx}(\lambda^*,x^*)\phi^*\phi^*)$,

and R_3 is twice continuously differentiable with

$$R_3(\delta\lambda,\varepsilon) = o\left((|\delta\lambda| + |\varepsilon|)^2 \right). \tag{4.1.6}$$

Thus we wish to find the small solutions of (4.1.5) which are identified with the solutions of (1.1) near (λ^*,x^*) by (4.1.2), and the equations $\lambda = \lambda^* + \delta\lambda$, $x = x^* + \varepsilon\phi^* + w(\delta\lambda,\varepsilon)$. To determine these solutions we need one final condition.

Definition 4.1

If, in addition to (4.1.1),

$$B^2 - 4AC > 0 \tag{4.1.7}$$

then (λ^*,x^*) is a SIMPLE bifurcation point .

Assuming (4.1.7) we now show that the solution-set of (1.1) near (λ^*,x^*) consists of two distinct curves, crossing locally only at (λ^*,x^*). The possibilities which can occur, if (4.1.7) does not hold, will be noted later. To prove the above statement we proceed in a manner similar to Bartle [3], but we do not need his assumptions that $A \neq 0$ and $C \neq 0$. Nor is it assumed a priori that there is a solution curve passing through (λ^*,x^*). The proof is split into three theorems (i) $C \neq 0$ (ii) $A \neq 0$ (iii) $A = C = 0$.

Theorem 4.2

If (λ^*, x^*) is a simple bifurcation point and $C \neq 0$, with p_i $i=1,2$ denoting the two distinct real roots of

$$C p^2 + Bp + A = 0 , \tag{4.1.8}$$

then a) the solution-set of (1.1) near (λ^*, x^*) can be parametrised by $\delta\lambda$ and consists of two distinct curves

$$x_i^*(\delta\lambda) = x^* + p_i(\delta\lambda)\,\delta\lambda\,\phi^* + w(\delta\lambda, p_i(\delta\lambda)\,\delta\lambda) \tag{4.1.9}$$

where $p_i(\delta\lambda) = p_i + o(1)$

b) $G_x(\lambda^* + \delta\lambda, x_i^*(\delta\lambda))$ is non-singular, for sufficiently small non-zero $|\delta\lambda|$ with

$$\| G_x(\lambda^* + \delta\lambda, x_i^*(\delta\lambda))^{-1} \| = O(|\delta\lambda|^{-1}) . \tag{4.1.10}$$

Proof

Choose $K > \max(|p_1|, |p_2|)$, and define a change of variable $(\delta\lambda, \varepsilon) \to (\delta\lambda, \eta)$ by means of the transformation

$$\eta = \begin{cases} 0 & \text{if } \delta\lambda = 0 \\ \varepsilon/\delta\lambda & \delta\lambda \neq 0 \end{cases} . \tag{4.1.11}$$

This defines an isomorphism between the region D_1, $|\delta\lambda| < \delta$ $|\varepsilon| < K|\delta\lambda|$, and the region D_2, $|\delta\lambda| < \delta$ $|\eta| < K$ minus $(0,\eta)$ $\eta \neq 0$.
The regions are shown in the diagram below.

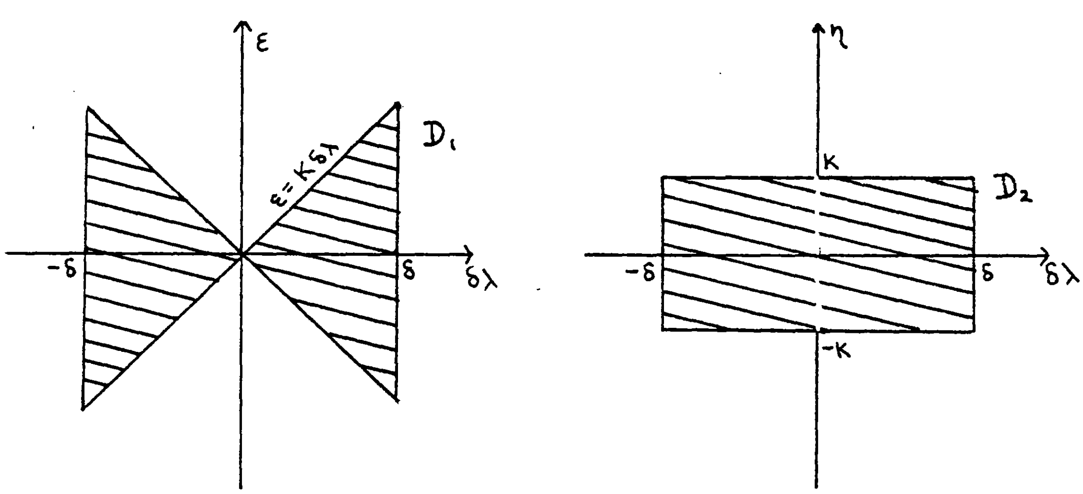

Fig. 4.1 D_1 in $(\delta\lambda, \varepsilon)$ plane and D_2 in $(\delta\lambda, \eta)$ plane.

Thus all the solutions $(\delta\lambda, \varepsilon)$ of (4.1.5) in D_1, will be mirrored by solutions $(\delta\lambda, \eta)$ in D_2, of the equation

$$A\delta\lambda^2 + B\eta\delta\lambda^2 + C\eta^2\delta\lambda^2 + R_3(\delta\lambda, \eta\delta\lambda) = 0 \ . \tag{4.1.12}$$

We disregard the solutions corresponding to $\delta\lambda = 0$, and divide through by $\delta\lambda^2$ to give

$$A + B\eta + C\eta^2 + R_4(\delta\lambda, \eta) = 0 \ , \tag{4.1.13}$$

where R_4 is continuous in $\delta\lambda$ and twice continuously differentiable in η, and $R_4(0,\eta) = 0$ for all η. All the solutions of (4.1.13) in D_2 will also be solutions in $\bar{D}_2$ (the closure of D_2) and thus we can insert the η- axis in Fig. 4.1b. Then, using the implicit function theorem, and restricting δ if necessary, we see that the only solutions are $\eta_i(\delta\lambda) = \rho_i + o(1)$ $i = 1,2$. These can be transformed back via (4.1.11) to give (4.1.9).

We now show that for K sufficiently large and $|\delta\lambda|$ sufficiently small, all the small solutions of (4.1.5) are in D_1. To demonstrate this it is sufficient to prove that $\varepsilon/\delta\lambda$ is bounded for such solutions. Firstly, if $\delta\lambda = 0$, the fact that $C \neq 0$ and R_3 is of higher order in (4.1.5) implies that no small non-zero ε -solution exists, thus we may assume that $\delta\lambda \neq 0$. Then dividing through (4.1.5) by $(|\varepsilon| + |\delta\lambda|)^2$ gives

$$\begin{aligned} -R_4(\delta\lambda, \varepsilon)/(|\varepsilon| + |\delta\lambda|)^2 &= (A\,\delta\lambda^2 + B\delta\lambda\varepsilon + C\varepsilon^2)/(|\varepsilon| + |\delta\lambda|)^2 \\ &= (A + B\alpha + C\alpha^2)/(1 + |\alpha|)^2 \ , \end{aligned} \tag{4.1.14}$$

where $\alpha = \varepsilon/\delta\lambda$. Now we may assume $|\alpha| > 1$, or else we already have a bound, and this gives

$$-R_4(\delta\lambda, \varepsilon)/(|\varepsilon| + |\delta\lambda|)^2 = C + K(\alpha) \ , \tag{4.1.15}$$

where $K(\alpha) = O(|\alpha|^{-1})$. If $\alpha = \varepsilon/\delta\lambda$ is unbounded the right-hand side of (4.1.15) can be made as close to C as desired, but this contradicts the fact that the left-hand side tends to zero as $\delta\lambda, \varepsilon \to 0$. Thus $\varepsilon/\delta\lambda$ must remain bounded and the only solutions are those by (4.1.9).

Proceeding to the second part of the theorem, by straight differentiation $G_x(\lambda^*+\delta\lambda, x_i^*(\delta\lambda))$ is given by

$$G_x(\lambda^*, x^*) + \delta\lambda\{G_{x\lambda}(\lambda^*, x^*) + G_{xx}(\lambda^*, x^*)(p_i\phi^* + w^*)\} + L(\delta\lambda) \quad (4.1.16)$$

where $\|L(\delta\lambda)\| = o(|\delta\lambda|)$. Because

$$\psi^*(G_{x\lambda}(\lambda^*, x^*)\phi^* + G_{xx}(\lambda^*, x^*)(p_i\phi^* + w^*)\phi^*) \quad (4.1.17)$$
$$= 2p_iC + B$$

is non-zero by (4.1.7) and (4.1.8), the result follows immediately from Lemma 2.1 and the Banach perturbation theorem ■

We now consider the case $A \neq 0$.

Theorem 4.3

If (λ^*, x^*) is a simple bifurcation point and $A \neq 0$, with q_i $i=1,2$ denoting the two distinct real roots of

$$Aq^2 + Bq + C = 0, \quad (4.1.18)$$

then a) the solution-set of (1.1) near (λ^*, x^*) may be parametrised by ε and consists of two distinct curves

$$\text{(i)} \quad \lambda_i^*(\varepsilon) = \lambda^* + q_i(\varepsilon)\varepsilon$$
$$\text{(ii)} \quad x_i(\varepsilon) = x^* + \varepsilon\phi^* + w(q_i(\varepsilon)\varepsilon, \varepsilon) \quad (4.1.19)$$

where $q_i(\varepsilon) = q_i + o(1)$ $i=1,2$,

b) $G_x(\lambda_i^*(\varepsilon), x_i^*(\varepsilon)) + G_\lambda(\lambda_i^*(\varepsilon), x_i^*(\varepsilon))$, regarded as an operator from R x N to X, has a bounded inverse, for sufficiently small non-zero ε, and

$$\|\{G_x(\lambda_i^*(\varepsilon), x_i^*(\varepsilon)) + G_\lambda(\lambda_i^*(\varepsilon), x_i^*(\varepsilon))\}^{-1}\| = O(|\varepsilon|^{-1}). \quad (4.1.20)$$

Proof

Part a) is proved in a similar way to Theorem 4.2 a) but using the change of variable $(\delta\lambda, \varepsilon) \to (\eta, \varepsilon)$ given by

$$\eta = \begin{cases} \delta\lambda/\varepsilon & \text{if } \varepsilon \neq 0 \\ 0 & \text{if } \varepsilon \neq 0 \end{cases} \quad (4.1.21)$$

and restricting attention to the region $|\varepsilon|<\delta$, $|\delta\lambda|<K|\varepsilon|$.

Part b) also follows in a similar way to Theorem 4.2 b) by noting that the linear operator can be written in the form

$$G_x(\lambda^*,x^*)+G_\lambda(\lambda^*,x^*)+\varepsilon\{G_{x\lambda}(\lambda^*,x^*)q_i \tag{4.1.22}$$
$$+G_{xx}(\lambda^*,x^*)(\phi^*+q_iw^*)+G_{\lambda\lambda}(\lambda^*,x^*)q_i+G_{\lambda x}(\lambda^*,x^*)(\phi^*+q_iw^*)\}+L(\varepsilon)$$

where $\|L(\varepsilon)\|=o(|\varepsilon|)$. As $\lambda=1$ and $x=w^*$ spans the null-space of $G_x(\lambda^*,x^*)+G_\lambda(\lambda^*,x^*)$, and

$$\psi^*(q_iG_{x\lambda}(\lambda^*,x^*)w^*+G_{xx}(\lambda^*,x^*)w^*(\phi^*+q_iw^*) \tag{4.1.23}$$
$$+q_iG_{\lambda\lambda}(\lambda^*,x^*)+G_{\lambda x}(\lambda^*,x^*)(\phi^*+q_iw^*))$$
$$=2q_iA+B\ ,$$

which is non-zero by (4.1.7) and (4.1.15), the result follows from Lemma 2.1 and the Banach perturbation theorem ∎

Looking at the solutions for $\delta\lambda$ in terms of ε given by Theorem 4.3 we see that at least one q_i, say q_1, must be non-zero, (if they are both non-zero then $C\neq 0$ and Theorem 4.2 applies). Thus the solution curve $(\lambda_1^*(\varepsilon),x_1^*(\varepsilon))$ can be parametrised by $\delta\lambda$ by simply using the implicit function theorem on the definition of $\delta\lambda$ given in terms of ε in 4.1.19. Also along this curve $G_x(\lambda,x)$ is non-singular for sufficiently small $|\lambda-\lambda^*|$. It can be written in the form

$$G_x(\lambda^*,x^*)+\varepsilon\{G_{xx}(\lambda^*,x^*)(\phi^*+q_1w^*)+G_{x\lambda}(\lambda^*,x^*)q_1\}+L(\varepsilon) \tag{4.1.24}$$

where $\|L(\varepsilon)\|=o(|\varepsilon|)$, and as

$$\psi^*(G_{xx}(\lambda^*,x^*)(\phi^*+q_1w^*)+G_{x\lambda}(\lambda^*,x^*)q_1) \tag{4.1.25}$$
$$=2C+q_1B$$
$$=-q_1(B^2-4AC)^{1/2}\neq 0\ ,$$

we can apply Lemma 2.1 and the Banach perturbation theorem. A parallel result holds for $C\neq 0$ and thus in either case we can parametrise one curve by $\delta\lambda$ and the other by ε.

Finally we deal with the case $A = C = 0$.

Theorem 4.4

If (λ^*, x^*) is a simple bifurcation point with both A and C zero, and hence B non-zero, the solution set of (1.1) near (λ^*, x^*) consists of two distinct curves; one of which may be parametrised by $\delta\lambda$ and written in the form

$$x_1^*(\delta\lambda) = x^* + \varepsilon(\delta\lambda)\phi^* + w(\delta\lambda, \varepsilon(\delta\lambda)) \qquad (4.1.26)$$

where $\varepsilon(\delta\lambda) = o(|\delta\lambda|)$, and the other which may be parametrised by ε and written in the form

$$\text{(i)} \quad \lambda_2^*(\varepsilon) = \lambda^* + \delta\lambda(\varepsilon)$$
$$\text{(ii)} \quad x_2^*(\varepsilon) = x^* + \varepsilon\phi^* + w(\delta\lambda(\varepsilon), \varepsilon) \qquad (4.1.27)$$

where $\delta\lambda(\varepsilon) = o(|\varepsilon|)$.

$G_x(\lambda^*+\delta\lambda, x_1^*(\delta\lambda))$ also satisfies part b) of Theorem 4.2 and $G_x(\lambda^*+\delta\lambda(\varepsilon), x_2^*(\varepsilon)) + G_\lambda(\lambda^*+\delta\lambda(\varepsilon), x_2^*(\varepsilon))$ part b) of Theorem 4.3.

Proof

The theorem is proved in exactly the same way as the previous two theorems. First using (4.1.11) to obtain (4.1.2.6) and the (4.1.21) to obtain (4.1.27)∎

Thus in all three cases we have one curve which can be parametrised by $\delta\lambda$ and another which can be parametrised by ε.

It is interesting to see how the results of this section tie up with bifurcation from the trivial solution in chapter 2. In the latter case $x^* = 0$ and $G_\lambda(\lambda^*, 0)$, $G_{\lambda\lambda}(\lambda^*, 0)$ and w^* are all identically zero. This gives

$$\begin{aligned} &\text{a)} \quad A = 0 \\ &\text{b)} \quad B = \psi^*(G_{x\lambda}(\lambda^*,0)\,\phi^*) \qquad\qquad (4.1.28) \\ &\text{c)} \quad C = \tfrac{1}{2}\psi^*(G_{xx}(\lambda^*,0)\,\phi^*\phi^*)\,. \end{aligned}$$

Thus (4.1.7), our condition for a simple bifurcation point, reduces to $B \neq 0$ which is identical to (2.1.9). Similarly the condition $C \neq 0$ for parametrisation by $\delta\lambda$ is identical to (2.6.7) for $p = 2$.

To conclude this section consider what may happen if (4.1.7) does not hold, with illustrative examples for $X = R$. If $B^2 - 4AC < 0$ then clearly A and C are non-zero and the theory of Newton's Polygon [3] asserts that (λ^*, x^*) is an isolated solution of (1.1) in R x X.

Example: $(x-1)^2 + (\lambda-1)^2 = 0$

for which $\lambda^* = x^* = 1$ is the only solution. If we are already following a solution curve then, as far as we are concerned, such points cannot occur. If $B^2 - 4AC = 0$ then a variety of types of behaviour are possible and we note the following examples .

a) (λ^*, x^*) is an isolated solution of (1.1)

Example: $(x-\lambda)^2 + (x-1)^4 + (\lambda-1)^4 = 0$

b) there is a unique solution curve passing through (λ^*, x^*) but each point of the curve satisfies (4.1.1)

Example: $(x-\lambda)^2 = 0$

c) there are two distinct solution curves passing through (λ^*, x^*) but they are tangential there.

Example: $(x^2+\lambda^2-1)(\lambda-1) = 0$

d) two solution curves coalesce at (λ^*, x^*)

Example: $(x - e^{1/\lambda})(x + e^{1/\lambda}) = 0 \qquad \lambda < 0$

$x = 0 \qquad \lambda \geqslant 0$

e) there are many solution curves passing through (λ^*, x^*)

Example: $\sum_{i=1}^{n} (\alpha_i \lambda + \beta_i x) = 0$

The rest of this chapter is ordered in the following manner. In section 4.2 we present a method for the determination of simple bifurcation points, similar to the approach in section 3.4 for turning points. The problems of numerical implementation are discussed in section 4.3. Finally, in section 4.4, we show how one may move away from the bifurcation point, along either of the solution curves, by methods analogous to those in chapter 2. Numerical results are given in section 4.5

4.2 Computing a Simple Bifurcation Point: Theory

For easier reference we first collect results about the singular point (λ^*, x^*), some of which have been used in earlier chapters, (see Vainberg and Trenogin [30, §21] although the notation is slightly different.)

Let l^* be the unique bounded linear functional on X such that $l^*(\phi^*)=1$ and $l^*(N)=0$, and Q the projection of X onto N defined by

$$Qx = x - l^*(x)\phi^*. \tag{4.2.1}$$

Let z^* be a unit element of X such that $\psi^*(z^*)=1$, and P the projection of X onto $R\{G_x(\lambda^*,x^*)\}$ defined by

$$Px = x - \psi^*(x)z^*. \tag{4.2.2}$$

M is the inverse of $G_x(\lambda^*,x^*)$ when this operator is restricted to N.

Using l^* and z^* we may also define projections in X′ (the dual of X). Thus, Q_1 is the projection of X′ onto $R\{G_x(\lambda^*,x^*)'\}$ defined by

$$Q_1 x' = x' - x'(\phi^*)l^*, \tag{4.2.3}$$

and P_1 the projection of X′ onto N_1, the annihilators of z^*, defined by

$$P_1 x' = x' - x'(z^*)\psi^*. \qquad (4.2.4)$$

$G_x(\lambda^*, x^*)'$, considered as an operator from N_1 to $\mathcal{R}\{G_x(\lambda^*, x^*)'\}$, also has a bounded inverse M_1.

Now we can define our method for computing a simple bifurcation point, namely to solve the system of equations

$$\begin{aligned} &\text{a)} \quad TG(\lambda, x) = 0 \\ &\text{b)} \quad G_x(\lambda, x)'\psi = 0 \\ &\text{c)} \quad \psi(G_\lambda(\lambda, x)) = 0 \\ &\text{d)} \quad \psi(v) - 1 = 0 \end{aligned} \qquad (4.2.5)$$

for $\lambda \in R$, $x \in X$ and $\psi \in X'$ by the Newton-Kantorovich method. Here T is a chosen bounded projection on X with 1-dimensional null-space, and v the unit element of $\mathcal{N}\{T\}$ unique up to sign. v and T can be thought of as approximations to z^* and P, which of course we do not know, and, so long as $\psi^*(v)$ is non-zero, a solution of (4.2.5) is given by λ^*, x^* and $\bar{\psi}^*$, where

$$\bar{\psi}^* = \psi^*/\psi^*(v). \qquad (4.2.6)$$

The idea behind (4.2.5) is that at a bifurcation point, as distinct from a non-singular point or a turning point, $G_x(\lambda^*, x^*) + G_\lambda(\lambda^*, x^*)$, considered as an operator from R x X to X, has a two-dimensional null-space spanned by $(0, \phi^*)$ and $(1, w^*)$. So instead we consider the dual operator $\left(G_x(\lambda^*, x^*) + G_\lambda(\lambda^*, x^*)\right)'$ from X' to $(R \times X)'$ which has a one-dimensional null-space spanned by ψ^*. Thus equations (4.2.5b) and 4.2.5c) replace equation (3.4.1b) in the determination of a simple turning point and (4.2.5d) is just a scaling equation for ψ. However we will then have an over-determined system unless, for the equation $G(\lambda, x) = 0$, G is regarded as an operator from R x X to a subspace of X of co-dimension 1. The natural choice for this subspace is $\mathcal{R}\{G_x(\lambda^*, x^*)\}$ or an approximation to it, and so we introduce T and arrive at (4.2.5a). It is also possible to set up equations for a simple bifurcation point without using dual operators and this is noted at the end of the section.

In order to analyse (4.2.5) we define $y = (\lambda, x, \psi)$ and re-write it as

$$F(y) = 0 \qquad F: Y_1 \rightarrow Y_2 \tag{4.2.7}$$

where $Y_1 = R \times X \times X$ and $Y_2 = TX \times X \times R \times R$. Both Y_1 and Y_2 become Banach spaces under the norms

$$\begin{aligned} &a) \quad \|y_1\|_{Y_1} = \max\{|\lambda|, \|x\|, \|x'\|\} \qquad y_1 = (\lambda, x, x') \\ &b) \quad \|y_2\|_{Y_2} = \max\{\|z\|, \|z'\|, |\mu_1|, |\mu_2|\} \qquad y_2 = (z, z', \mu_1, \mu_2) \end{aligned} \tag{4.2.8}$$

and, of course, they are linearly homeomorphic, but we do not use this.

From now on we do not distinguish between the different norms but let this be deduced from the particular element being normed.

We can now prove the following important lemma about the system (4.2.5).

Lemma 4.5

If (λ^*, x^*) is a simple bifurcation point and

$$\psi^*(v) \neq 0 \tag{4.2.9}$$

then a) $TG_x(\lambda^*, x^*)$ has a bounded inverse from TX to SX, where S is any bounded projection on X with 1-dimensional null-space and $(I-S)\phi^* \neq 0$.

b) $F_y(y^*)$ has a bounded inverse from Y_2 to Y_1, where $y^* = (\lambda^*, x^*, \bar{\psi}^*)$.

Proof

For the first part of the theorem, given $y \in TX$, we shall solve

$$TG_x(\lambda^*, x^*)\, x = y \tag{4.2.10}$$

for $x \in SX$. This is equivalent to

$$G_x(\lambda^*, x^*)\, x = \alpha v + y \tag{4.2.11}$$

for some $\alpha \in R$, but, for $\alpha v + y$ to be in $\mathcal{R}\{G_x(\lambda^*, x^*)\}$, we must have $\alpha = -\psi^*(y)/\psi^*(v)$. Then the solution of (4.2.11) is given by

$$x = \beta\phi^* + M\{y - \psi^*(y)v/\psi^*(v)\} \tag{4.2.12}$$

for arbitrary β. However, for $x \in SX$, β is determined uniquely by

$$\beta(I-S)\phi^* = -(I-S)M\{y - \psi^*(y)v/\psi^*(v)\}\ . \tag{4.2.13}$$

The above construction shows that $\mathcal{R}\{TG_x(\lambda^*, x^*)\} = TX$ and $\mathcal{N}\{TG_x(\lambda^*, x^*)\} = \{0\}$ in SX and thus, by the open-mapping theorem, $G_x(\lambda^*, x^*)$ has a bounded inverse from TX to SX.

For the second part of the theorem we proceed in a similar manner. If $F_y(y^*)y_1 = y_2$, with $y_1 = (\lambda, x, x')$ and $y_2 = (z, z', \mu_1, \mu_2)$, then, from the definition of F,

$$\begin{aligned}
&\text{(i)} \quad TG_x(\lambda^*, x^*)x + TG_\lambda(\lambda^*, x^*)\lambda = z \\
&\text{(ii)} \quad G_x(\lambda^*, x^*)'x' + (G_{xx}(\lambda^*, x^*)x)'\bar{\psi}^* \\
&\qquad\qquad + (G_{x\lambda}(\lambda^*, x^*)\lambda)'\bar{\psi}^* = z' \\
&\text{(iii)} \quad x'(G_\lambda(\lambda^*, x^*)) + \bar{\psi}^*(G_{\lambda\lambda}(\lambda^*, x^*)\lambda + G_{\lambda x}(\lambda^*, x^*)x) = \mu_1 \\
&\text{(iv)} \quad x'(v) = \mu_2\ .
\end{aligned} \tag{4.2.14}$$

Using the projections defined at the beginning of this section together with part a) of the theorem with S = Q, gives, after simple manipulation, the equivalent form

(i) $Qx = \lambda w^* + Lz$

where $Lz = M\{z - \psi^*(z)v/\psi^*(v)\}$

(ii)a $P_1 x' = M_1\{z' - (G_{xx}(\lambda^*, x^*)x)'\bar{\psi}^* - (G_{x\lambda}(\lambda^*, x^*)\lambda)'\bar{\psi}^*\}$

$$(ii)b. \quad \lambda\bar{\psi}^*(G_{xx}(\lambda^*,x^*)w^*\phi^* + G_{x\lambda}(\lambda^*,x^*)\phi^*) + l^*(x)\bar{\psi}^*(G_{xx}(\lambda^*,x^*)\phi^*\phi^*) = z'(\phi^*) - \bar{\psi}^*(G_{xx}(\lambda^*,x^*)\phi^* Lz) \tag{4.2.15}$$

$$(iii) \quad \lambda\bar{\psi}^*(G_{\lambda\lambda}(\lambda^*,x^*) + 2G_{\lambda x}(\lambda^*,x^*)w^* + G_{xx}(\lambda^*,x^*)w^*w^*) + l^*(x)\bar{\psi}^*(G_{\lambda x}(\lambda^*,x^*)\phi^* + G_{xx}(\lambda^*,x^*)\phi^*w^*) = \mu_1 + z'(w^*) - \psi^*(G_{\lambda x}(\lambda^*,x^*)Lz + G_{xx}(\lambda^*,x^*)w^*Lz)$$

$$(iv) \quad x'(z^*) = \{\mu_2 - (P_1x')v\}/\psi^*(v).$$

Using (4.2.6) the two simultaneous equations for λ and $l^*(x)$, (4.2.15) (ii) b) and (iii), can be written

$$\begin{aligned} &a) \quad B\lambda + 2C\,l^*(x) = d_1 \\ &b) \quad 2A\lambda + B\,l^*(x) = d_2 \end{aligned} \tag{4.2.16}$$

and, as the determinant of this system is B^2-4AC, which is non-zero by (4.1.7), λ and $l^*(x)$ are uniquely determined in terms of the other variables. Qx is then given by (4.2.15) (i) and the components of x' by (ii) a) and (iv). Thus, by construction, F_y is 1-1 and onto Y_2 and so, by the open-mapping theorem, part b) of the theorem is proved ∎

Now that we know $F(y^*)=0$ and $F_y(y^*)^{-1}$ exists, and thus y^* is an isolated solution of (4.2.7), Theorem 2.3 can be applied to prove the convergence of Newton's method.

Theorem 4.6

If (λ^*,x^*) is a simple bifurcation point, with T chosen so that (4.2.9) holds, and the second-order partial derivatives of G satisfy a uniform Lipschitz condition for $|\lambda-\lambda^*|\leq\tau_1$ and $\|x-x^*\|\leq\tau_2$ then there exists $0<\tau<\tau_1$ such that, if $\|y^0-y^*\|\leq\tau$ the Newton iteration for (4.2.7)

$$y^{n+1} = y^n - F_y(y^n)^{-1}F(y^n) \qquad n=0,1,2\ldots \tag{4.2.17}$$

converges quadratically to y^*.

Proof

The theorem and proof are very similar to Theorem 3.4. Let $y^\circ = (\lambda^\circ, x^\circ, \psi^\circ)$, then expanding about y^* gives

$$\begin{aligned}
&\text{(i)} \quad \|TG(\lambda^\circ,x^\circ)\| \leq \|T\|\{\|G_\lambda(\lambda^*,x^*)\|\,|\lambda-\lambda^*| + \|G_x(\lambda^*,x^*)\|\,\|x^\circ-x^*\| + 2K_1\tau^2\} \\
&\text{(ii)} \quad \|G_x(\lambda^\circ,x^\circ)'\psi^\circ\| \leq \|G_x(\lambda^*,x^*)'\|\,\|\psi^\circ-\bar{\psi}^*\| + 2K_1\tau(a+\tau) \\
&\text{(iii)} \quad |\psi^\circ(G_\lambda(\lambda^\circ,x^\circ))| \leq \|G_\lambda(\lambda^*,x^*)\|\,\|\psi^\circ-\bar{\psi}^*\| + 2K_1\tau(a+\tau) \qquad (4.2.18) \\
&\text{(iv)} \quad |\psi^\circ(v)-1| \leq \tau
\end{aligned}$$

where $a = \|\bar{\psi}^*\|$ and K_1 is a uniform bound on the second derivatives of G. Thus there exists $d_1 \geq 0$ such that

$$\|F(y^\circ)\| \leq d_1\tau . \qquad (4.2.19)$$

$F_y(y^*)^{-1}$ exists, from Lemma 4.5, with

$$\|F_y(y^*)^{-1}\| \leq d_2 \qquad (4.2.20)$$

and, if K_2 is the uniform Lipschitz constant of the theorem, simple algebra, like (4.2.18), gives

$$\|F_y(y^\circ) - F_y(y^*)\| \leq d_3\|y^\circ - y^*\| \qquad (4.2.21)$$

where $d_3 = \max\{4K_1\|T\|, 4(K_1+K_2a)\}$. Thus if τ is restricted so that $\delta_\tau = 1-d_2d_3\tau > 0$, $F_y(y^\circ)^{-1}$ will exist, by the Banach perturbation theorem, with

$$\|F_y(y^\circ)^{-1}\| \leq d_2\delta_\tau^{-1}. \qquad (4.2.22)$$

Using the notation of Theorem 2.3, $\eta = d_1d_2\delta_\tau^{-1}\tau$, and so, to obtain a Lipschitz condition for $F_y(y)$ in the ball $\|y-y^\circ\| < 2\eta$, we must restrict τ so that

$$\tau(1+2d_1d_2\delta_\tau^{-1}) \leq \tau_1 . \qquad (4.2.23)$$

With τ so restricted, $\|z_i - y^\circ\| \leq 2\eta \quad i = 1,2$ implies

$$\|F_y(z_1) - F_y(z_2)\| \leq d_3\|z_1-z_2\| , \qquad (4.2.24)$$

and thus h of Theorem 2.3 satisfies $h \leq d_1d_3d_2^2\delta_\tau^{-2}\tau$, and τ restricted so that $h < \frac{1}{2}$ proves the theorem∎

If the second derivatives of G are merely continuous, and do not satisfy a Lipschitz condition, the Newton iterates still converge, for sufficiently close starting values, but quadratic convergence cannot be asserted (Krasnosel'skii [38, p. 140]).

Finally we note that it is also possible to compute a simple bifurcation point by a method similar to (4.2.5) but which avoids the use of dual operators. Instead of determining an element $\bar{\psi}^*$ which spans the null-space of $(G_x(\lambda^*,x^*)+G_\lambda(\lambda^*,x^*))'$, we determine two independent elements $\bar{\phi}^*$ and $\bar{w}^*$ which span the null-space of $G_x(\lambda^*,x^*)$ $+G_\lambda(\lambda^*,x^*)$ i.e. we solve the system

$$
\begin{aligned}
&\text{(i)} \quad TG(\lambda,x)=0\\
&\text{(ii)} \quad G_x(\lambda,x)\phi=0\\
&\text{(iii)} \quad G_x(\lambda,x)w+G_\lambda(\lambda,x)=0 \qquad\qquad (4.2.25)\\
&\text{(iv)} \quad l(\phi)-1=0\\
&\text{(v)} \quad l(w)=0
\end{aligned}
$$

for $\lambda \in R$ and $x,\phi,w \in X$. Here T is as in (4.2.5) and l is a scaling bounded linear functional which may be regarded as an approximation to l^*. So long as $\psi^*(v)$ and $l(\phi^*)$ are non-zero a result similar to Lemma 4.5 can be obtained and Newton's method will converge. However this method is generally less efficient as is seen in the next section.

4.3 Computing a Simple Bifurcation Point: Implementation

In this section, proceeding in a similar manner to section 3.5, we develop an efficient algorithm for the numerical solution of the linear systems which arise when the Newton iteration (4.2.17) is carried out in R^n, after an approximation $(\lambda^\circ, \underset{\sim}{x}^\circ)$ to the bifurcation point has been obtained by a continuation method.

In R^n, $G_x(\lambda, \underset{\sim}{x})'$ can be identified with $G_x(\lambda, \underset{\sim}{x})^T$ and approximations $\underset{\sim}{\phi}^{\circ}$ and $\underset{\sim}{\psi}^{\circ}$ to $\underset{\sim}{\phi}^*$ and ψ^* can be obtained by inverse iteration applied to $G_x(\lambda_o, \underset{\sim}{x}^{\circ})$ and $G_x(\lambda^{\circ}, \underset{\sim}{x}^{\circ})^T$ respectively. Alternatively, if $G_x(\lambda, \underset{\sim}{x})$ is symmetric and the continuation method is based on Newton's method, Simpson's idea, mentioned briefly in section 3.5, can be used. We assume that $\underset{\sim}{\phi}^{\circ}$ and $\underset{\sim}{\psi}^{\circ}$ are normalised so that their largest components in modulus, ϕ_s° and ψ_t° are unity. Now we can define suitable T and $\underset{\sim}{v}$ for (4.2.5) and a suitable S for part a) of Lemma 4.5.

$$
\begin{aligned}
&\text{(i)} \quad \underset{\sim}{v} = \underset{\sim}{e}_t \\
&\text{(ii)} \quad T\underset{\sim}{x} = \underset{\sim}{x} - (\underset{\sim}{e}_t^T \underset{\sim}{x})\underset{\sim}{e}_t \qquad (4.3.1) \\
&\text{(iii)} \quad S\underset{\sim}{x} = \underset{\sim}{x} - (\underset{\sim}{e}_s^T \cdot \underset{\sim}{x})\underset{\sim}{e}_s
\end{aligned}
$$

where $\underset{\sim}{e}_s$ and $\underset{\sim}{e}_t$ are the s^{th} and t^{th} columns respectively of I_n, the identity matrix in R^n. Thus S/T maps the s/tth component of $\underset{\sim}{x}$ to zero.

$\underset{\sim}{y}^{\circ} = (\lambda^{\circ}, \underset{\sim}{x}^{\circ}, \underset{\sim}{\psi}^{\circ})$ is the starting value for (4.2.17) and, if the conditions of Theorem 4.6 hold, the Newton iteration

$$
\begin{aligned}
&\text{a)} \quad F_y(\underset{\sim}{y}^*)\, \delta\underset{\sim}{y}^n = -F(\underset{\sim}{y}^n) \\
&\text{b)} \quad \underset{\sim}{y}^{n+1} = \underset{\sim}{y}^n + \delta\underset{\sim}{y}^n
\end{aligned} \qquad (4.3.2)
$$

will converge to $\underset{\sim}{y}^* = (\lambda^*, \underset{\sim}{x}^*, \underset{\sim}{\psi}^*)$. If we suppress the superscripts on $\delta\underset{\sim}{y} = (\delta\lambda, \delta\underset{\sim}{x}, \delta\underset{\sim}{\psi})$, and write (4.3.2) in terms of G then we have

$$
\begin{aligned}
&\text{(i)} \quad TA\delta\underset{\sim}{x} + \delta\lambda\, \underset{\sim}{d}_1 = T\underset{\sim}{c}_1 \\
&\text{(ii)} \quad A^T\delta\underset{\sim}{\psi} + B\delta\underset{\sim}{x} + \delta\lambda\, \underset{\sim}{d}_2 = \underset{\sim}{c}_2 \\
&\text{(iii)} \quad \underset{\sim}{d}_1^T \delta\underset{\sim}{\psi} + \underset{\sim}{d}_2^T \delta\underset{\sim}{x} + \alpha\,\delta\lambda = c_3 \\
&\text{(iv)} \quad \delta\psi_t = 0
\end{aligned} \qquad (4.3.3)
$$

where

$$
\begin{aligned}
A &= G_x(\lambda^n, \underset{\sim}{x}^n) & B &= \{G_{xx}(\lambda^n, \underset{\sim}{x}^n)\,.\,\}^T \underset{\sim}{\psi}^n \\
\underset{\sim}{d}_1 &= G_\lambda(\lambda^n, \underset{\sim}{x}^n) & \underset{\sim}{d}_2 &= G_{x\lambda}(\lambda^n, \underset{\sim}{x}^n)^T \underset{\sim}{\psi}^n \\
\alpha &= G_{\lambda\lambda}(\lambda^n, \underset{\sim}{x}^n)^T \underset{\sim}{\psi}^n & \underset{\sim}{c}_1 &= -G(\lambda^n, \underset{\sim}{x}^n) \\
\underset{\sim}{c}_2 &= -G_x(\lambda^n, \underset{\sim}{x}^n)^T \underset{\sim}{\psi}^n & c_3 &= -G_\lambda(\lambda^n, \underset{\sim}{x}^n)^T \underset{\sim}{\psi}^n .
\end{aligned}
$$

Introducing the projection S, (4.3.3) (i) and (ii) can be written

$$\begin{aligned}
&\text{a)}\quad TAS\,\delta\underset{\sim}{x} = T\underset{\sim}{c}_1 - \delta\lambda T\underset{\sim}{d}_1 - \delta x_s TA\underset{\sim}{e}_s \\
&\text{b)}\quad (TAS)^T\delta\underset{\sim}{\psi} = S\underset{\sim}{c}_2 - \delta\lambda S\underset{\sim}{d}_2 - SB\,\delta\underset{\sim}{x} \qquad (4.3.4) \\
&\text{c)}\quad \underset{\sim}{e}_s^T A^T\delta\underset{\sim}{\psi} + \underset{\sim}{e}_s^T B\,\delta\underset{\sim}{x} + \delta\lambda\,\underset{\sim}{e}_s^T\underset{\sim}{d}_2 = \underset{\sim}{e}_s^T\underset{\sim}{c}_2
\end{aligned}$$

using (4.3.3) (iv) and the symmetry of S and T. The zero elements introduced by S and T into (4.3.4) a) and b) are eliminated by reducing to the $(n-1)\times(n-1)$ systems

$$\begin{aligned}
&\text{(i)}\quad \bar{A}\,\delta\underset{\sim}{u} = \underset{\sim}{f}_1 + \delta\lambda\,\underset{\sim}{f}_2 + \delta x_s\,\underset{\sim}{f}_3 \\
&\text{(ii)}\quad \bar{A}^T\delta\underset{\sim}{v} = \underset{\sim}{f}_4 + \delta\lambda\,\underset{\sim}{f}_5 + \bar{B}\,\delta\underset{\sim}{x}\,,
\end{aligned} \qquad (4.3.5)$$

where $\delta\underset{\sim}{u} = (\delta x_1, \ldots, \delta x_{s-1}, \delta x_{s+1}, \ldots, \delta x_n)^T$

$\delta\underset{\sim}{v} = (\delta\psi_1, \ldots, \delta\psi_{t-1}, \delta\psi_{t+1}, \ldots, \delta\psi_n)^T$

and $\bar{A}$ is A with s^{th} column and t^{th} row removed. $\bar{B}$ and the $\underset{\sim}{f}_i$ are related in an obvious way to the R.H.S. of (4.3.4).

If the conditions of Theorem 4.6 hold $\bar{A}$ will be non-singular and $\delta\underset{\sim}{u}$ can be found in terms of $\delta\lambda$ and δx_s from (4.3.5) (i). Inserting these values for $\delta x_i, i\neq s$, into (4.3.5) (ii) gives

$$\bar{A}^T\delta\underset{\sim}{v} = \underset{\sim}{g}_1 + \delta\lambda\,\underset{\sim}{g}_2 + \delta x_s\,\underset{\sim}{g}_3\,, \qquad (4.3.6)$$

which can similarly be solved to give $\delta\underset{\sim}{v}$ in terms of $\delta\lambda$ and δx_s. Finally $\delta x_i, i\neq s$, and $\delta\psi_i$ can be replaced in (4.3.3) (iii) and (4.3.4c) to give two simultaneous equations for $\delta\lambda$ and δx_s, whose determinant will be non-zero according to Theorem 4.6. Having obtained solutions for $\delta\lambda$ and δx_s these are inserted back into the expressions for $\delta\underset{\sim}{u}$ and $\delta\underset{\sim}{v}$ to give $\delta\underset{\sim}{x}$ and $\delta\underset{\sim}{\psi}$. This completes a single Newton iteration.

REMARK 1: The calculation of each new Newton iterate requires the solution of six $(n-1)\times(n-1)$ linear systems, (4.3.5) (i) and (4.3.6) three with coefficient matrix $\bar{A}$ and three with $\bar{A}^T$. Thus only one LU decomposition need be performed.

REMARK 2: If our matrix has convenient diagonal structure then this will be destroyed when a different row and column are removed to form $\bar{A}$. Usually, however, diagonal structure is accompanied by symmetry and hence the same row and column will be removed, retaining the structure. Even when symmetry is absent it may be possible to choose components ϕ_k^o and ψ_k^o so that

$$|\phi_k^o||\psi_k^o|/(\|\underline{\phi}^o\|_\infty\|\underline{\psi}^o\|_\infty)$$

is close to unity and both S and T can be defined using $\underline{e}_k$.

REMARK 3: At the end of the previous section we mentioned an alternative method for computing a simple bifurcation point, (4.2.25). The linear equations derived from applying Newton's method to this system may be solved in a very similar manner; each iteration consisting of solving a number of equations with coefficient matrix $\bar{A}$. However in this case nine $(n-1) \times (n-1)$ systems per iteration need to be solved and so, in general, the method described in this section should be preferred.

We also point out that REMARK 2 at the end of section 3.5 applies with equal force to the computation of bifurcation points.

4.4 Moving away from Simple Bifurcation Points

In this section we assume that a simple bifurcation point has already been computed and now we wish to continue along one of the solution curves emanating from it. We use the same general idea as that of chapter 2 and section 3.6 i.e. Corollary 2.4 is applied with starting values obtained from the Liapunov-Schmidt analysis.

Using (4.2.5) to determine the bifurcation point automatically gives ψ^*, and ϕ^* is the unique solution of

$$G_x(\lambda^*, x^*)\,\phi^* = 0 \qquad \|\phi^*\| = 1 . \tag{4.4.1}$$

Having obtained ϕ^* we can choose l^* and N, and then w^* is the unique solution of

$$\begin{aligned} &\text{a)} \quad G_x(\lambda^*, x^*)\,w^* = -G_\lambda(\lambda^*, x^*) \\ &\text{b)} \quad l^*(w^*) = 0 . \end{aligned} \tag{4.4.2}$$

Thus A, B and C of (4.1.5) can be calculated, if necessary by means of the approximations (3.5.20).

If $C \neq 0$, Theorem 4.2 shows that both solution curves may be parametrised by $\delta\lambda$ and written

$$x_i^*(\delta\lambda) = x^* + (p_i\phi^* + w^*)\,\delta\lambda + o(|\delta\lambda|), \tag{4.4.3}$$

where

$$p_i = \{-B + (-1)^i (B^2 - 4AC)^{1/2}\}/2C . \tag{4.4.4}$$

Thus both curves may be computed with λ held constant, as in section 2.6.

<u>Theorem 4.7</u>

For fixed $\delta\lambda$ sufficiently small the Newton iteration

$$x^{n+1}(\delta\lambda) = x^n(\delta\lambda) - G_x(\lambda^* + \delta\lambda, x^n(\delta\lambda))^{-1} G(\lambda^* + \delta\lambda, x^n(\delta\lambda)), \tag{4.4.5}$$

with starting approximation

$$x^0(\delta\lambda) = x^* + (p_i\phi^* + w^*)\,\delta\lambda , \tag{4.4.6}$$

converges to the solution curve $(\lambda^* + \delta\lambda, x_i^*(\delta\lambda))$.

<u>Proof</u>

We define

$$T(|\delta\lambda|, x) = G(\lambda^* + \delta\lambda, x) \tag{4.4.7}$$

and verify the conditions (i) to (iv) of Corollary 2.4.

$G_x(\lambda^*+\delta\lambda, x^\circ(\delta\lambda))$ is non-singular for sufficiently small non-zero $\delta\lambda$, by Theorem 4.2 part b), and

$$\|G_x(\lambda^*+\delta\lambda, x^\circ(\delta\lambda))^{-1}\| = O(|\delta\lambda|^{-1}), \tag{4.4.8}$$

but
$$\|G_x(\lambda^*+\delta\lambda, x^\circ(\delta\lambda))^{-1}x\| \leq K\|x\| \tag{4.4.9}$$

if $x \in \mathcal{R}\{G_x(\lambda^*, x^*)\}$. Thus condition (i) holds.

$$\begin{aligned} G(\lambda^*+\delta\lambda, x^\circ(\delta\lambda)) = \delta\lambda\,&\{G_x(\lambda^*, x^*)(p_i\phi^*+w^*) + G_\lambda(\lambda^*, x^*)\} \\ &+ \frac{\delta\lambda^2}{2}\{G_{xx}(\lambda^*, x^*)(p_i\phi^*+w^*)^2 + 2G_{x\lambda}(\lambda^*, x^*)(p_i\phi^*+w^*) \\ &\quad + G_{\lambda\lambda}(\lambda^*, x^*)\} + o(|\delta\lambda|^2) \end{aligned} \tag{4.4.10}$$

and so
$$\|G(\lambda^*+\delta\lambda, x^\circ(\delta\lambda))\| = O(|\delta\lambda|^2), \tag{4.4.11}$$

but
$$|\psi^*(G(\lambda^*+\delta\lambda, x^\circ(\delta\lambda)))| = o(|\delta\lambda|^2) \tag{4.4.12}$$

by definition of p_i. Combining (4.4.8), (4.4.9), (4.4.11) and (4.4.12) gives

$$\|G_x(\lambda^*+\delta\lambda, x^\circ(\delta\lambda))^{-1}G(\lambda^*+\delta\lambda, x^\circ(\delta\lambda))\| = K_1(\delta\lambda), \tag{4.4.13}$$

where $K_1(\delta\lambda) = o(|\delta\lambda|)$, and so condition (ii) holds.

If $\|z_i - x^\circ(\delta\lambda)\| < 2K_1(\delta\lambda)$ $i=1,2$ then, using

$$\begin{aligned} &G_x(\lambda^*+\delta\lambda, z_1) - G_x(\lambda^*+\delta\lambda, z_2) \\ &\quad = \int_0^1 G_{xx}(\lambda, z_2+t(z_1-z_2))(z_1-z_2)\,dt \;, \end{aligned} \tag{4.4.14}$$

we have

$$\begin{aligned} &\|G_x(\lambda^*+\delta\lambda, x^\circ(\delta\lambda))^{-1}\{G_x(\lambda^*+\delta\lambda, z_1) - G_x(\lambda^*+\delta\lambda, z_2)\}\| \\ &\qquad \leq K_2(\delta\lambda)\,\|z_1-z_2\|\,, \end{aligned} \tag{4.4.15}$$

where $K_2(\delta\lambda) = O(|\delta\lambda|^{-1})$, and so condition (iii) holds.

Finally, $K_1(\delta\lambda)K_2(\delta\lambda) = o(1)$ and so condition (iv) holds and the theorem is proved ∎

If $C = 0$, and therefore $B \neq 0$ from (4.1.7), only one of the solution curves can be parametrised by $\delta\lambda$. This is given by

$$x^*(\delta\lambda) = x^* + \delta\lambda(w^* - A\phi^*/B) + o(|\delta\lambda|) \tag{4.4.16}$$

and in this case the Newton iteration (4.4.5), with starting approximation $x^0(\delta\lambda) = x^* + \delta\lambda(w^* - A\phi^*/B)$, will converge just as in Theorem 4.7, and we omit the proof.

If $A \neq 0$, Theorem 4.3 shows that both solution curves may be parametrised by ϵ and written

$$\begin{aligned} &\text{(i)} \quad \lambda_i^*(\epsilon) = \lambda^* + q_i(\epsilon)\epsilon + o(|\epsilon|) \\ &\text{(ii)} \quad x_i^*(\epsilon) = x^* + \epsilon\phi^* + \epsilon q_i w^* + o(|\epsilon|) \end{aligned} \tag{4.4.17}$$

where

$$q_i = \{-B + (-1)^i (B^2 - 4AC)^{1/2}\}/2A \quad . \tag{4.4.18}$$

If the operator $H: R \times (R \times N) \longrightarrow X$, defined by

$$H(\epsilon, y) = G(\lambda, x^* + \epsilon\phi^* + w) \qquad y = (\lambda, w) \ , \tag{4.4.19}$$

is introduced we can prove an analogous theorem to Theorem 4.7, similar to those of section 2.5, and we omit the proof.

<u>Theorem 4.8</u>

For fixed ϵ sufficiently small the Newton iteration

$$y^{n+1}(\epsilon) = y^n(\epsilon) - H_y(\epsilon, y^n(\epsilon))^{-1} H(\epsilon, y^n(\epsilon)) \ , \tag{4.4.20}$$

with starting approximation

$$\begin{aligned} y^0(\epsilon) &= (\lambda(\epsilon), w(\epsilon)) \\ &= (\lambda^* + \epsilon q_i, \epsilon q_i w^*) \end{aligned} \tag{4.4.21}$$

converges to the solution curve $(\lambda_i^*(\epsilon), x_i^*(\epsilon))$.

If both A and C are zero then Theorem 4.4 shows that one curve must be parametrised by $\delta\lambda$, and is of the form

$$x^*(\delta\lambda) = x^* + o(|\delta\lambda|) \ , \tag{4.4.22}$$

and the other must be parametrised by ϵ and is of the form

$$\text{(i)} \quad \lambda^*(\epsilon) = \lambda^* + o(|\epsilon|) \tag{4.4.23}$$

(ii) $x^*(\varepsilon) = x^* + \varepsilon\phi^* + o(|\varepsilon|)$.

Newton's method with starting approximation

$$x^0(\delta\lambda) = x^* \tag{4.4.24}$$

can be used to obtain the first curve (4.4.22), and Newton's method with starting approximation

$$y^0(\varepsilon) = (\lambda^*, 0) \tag{4.4.25}$$

can be used to obtain the second curve (4.4.23).

Finally we mention that in [44] Keller has given notification of similar independent results. These use the idea of (3.2.11) and extend (1.1) to an equation from R x X to R x X,

$$\left.\begin{array}{ll} \text{a)} & \tilde{G}(y) = 0 \\ \text{b)} & \tilde{l}(y - y^*) - c = 0 \end{array}\right. \tag{4.4.26}$$

where $y = (\lambda, x)$ and $y^* = (\lambda^*, x^*)$. c is now the independent parameter and the choice of $\tilde{l} \in (R \times X)'$ determines which solution-curve is followed. By introducing the operator J: R x (R x X) $\rightarrow$ R x X defined by (4.4.26), and considering the equation

$$J(c, y) = 0 \tag{4.4.27}$$

this can be analysed in exactly the same way as previously in this chapter. We briefly state the results.

Let $z_i \in (R \times X)$ $i=1,2$ be the tangents to the two solution curves crossing at y^* i.e.

$$z_i = \begin{cases} (1, p_i\phi^* + w^*) & C \neq 0 \\ (q_i, \phi^* + q_i w^*) & A \neq 0 \\ (0, \phi^*) \text{ and } (1, w^*) & A = C = 0 \end{cases} \tag{4.4.28}$$

If we choose $\tilde{l}_j$ so that

$$\tilde{l}_j(z_i) = \delta_{ij}, \tag{4.4.29}$$

the Hahn-Banach theorem guarantees this can be done, then using $\tilde{l}_j$ in (4.4.27) will compute Γ_j with continuation by c. From now on we assume we are using $\bar{l}_1$.

The following statements are now easily verified

a) $(0, y^*)$ is a solution of (4.4.27)

b) $\mathcal{N}\{J_y(0,y^*)\}$ is 1-dimensional and spanned by z_2.

c) $\mathcal{N}\{J_y(0,y^*)'\}$ is 1-dimensional and spanned by $(0,\psi^*)$.

d) $J_c(0,y^*) = (-1,0)$ and

$$J_y(0,y^*)z_1 + J_c(0,y^*) = 0 \tag{4.4.30}$$

and z_1 takes the role of "w^*"

e) Computing (4.1.5) for $\varepsilon z_2 + c z_1$ reduces to calculating

$$(0,\psi^*)\left(J_{yy}(0,y^*)(\varepsilon z_2 + c z_1)\right) \tag{4.4.31}$$

as both J_{cc} and J_{cy} are zero. However as (4.4.31) is just

$$\psi^*\left(\left\{G_{xx}(\lambda^*,x^*) + 2G_{x\lambda}(\lambda^*,x^*) + G_{\lambda\lambda}(\lambda^*,x^*)\right\}(\varepsilon z_2 + c z_1)\right) \tag{4.4.32}$$

We have $A = C = 0$ in (4.1.5), and $\varepsilon = 0$ $c = 1$ and $\varepsilon = 1$ $c = 0$ give the solutions.

f) Thus continuation by c gives the Γ_1 solution-curve.

4.5 Numerical Results

Here and in the next chapter, we shall consider a generalisation of a class of problems in Keller [44]. These are non-linear boundary-value problems of the form

$$\text{a)}\quad x'' - q(\lambda)V''(t) + \pi^2\lambda\, p(x - q(\lambda)V(t)) = 0$$
$$\text{b)}\quad x(0) = x(1) = 0 \qquad (4.5.1)$$

where q, p and V are given functions with $V(0) = V(1) = 0$, $p(0) = 0$ and $p'(0) = 1$. Thus (4.5.1) has the solution

$$x^{\circ}(t,\lambda) = q(\lambda)V(t) \qquad (4.5.2)$$

for arbitrary λ. The linearisation of equation (4.5.1), about the solution (4.5.2), gives

$$\text{a)}\quad x'' + \pi^2\lambda x = 0$$
$$\text{b)}\quad x(0) = x(1) = 0 \qquad (4.5.3)$$

and so the singular points on the solution curve (4.5.2) are at

$$\lambda = k^2 \qquad k = 1, 2, \ldots \qquad (4.5.4)$$

In the example below we have chosen

$$\text{a)}\quad V(t) = t^2(1-t)$$
$$\text{b)}\quad p(z) = z + z^2 \qquad (4.5.5)$$
$$\text{c)}\quad q(\lambda) = \lambda^2 e^{-\lambda/2}$$

and used Euler-Newton continuation to follow the curve (4.5.2) from $\lambda = 0$ towards $\lambda = 1$. The simplest finite-difference approximation was used, cf. (2.8.3), with $h = 1/32$.

In Table 1 we show the convergence of our method to the bifurcation point after the continuation method had registered the presence of a singular point.

Table 1

λ	$\|\underset{\sim}{x}\|_2$	$\lvert\delta\lambda\rvert$	$\|\delta\underset{\sim}{x}\|_2$	$\|\delta\underset{\sim}{\phi}\|_2$
0.9123322075	0.2911891624			
		0.87 E-1	0.42 E-1	0.54 E-4
0.9995665259	0.3332859001			
		0.37 E-3	0.12 E-2	0.54 E-4
0.9991966876	0.3344330979			
		0.38 E-6	0.22 E-6	0.53 E-7
0.9991970675	0.3344329275			
		0.67 E-13	0.27 E-13	0.54 E-14
0.9991970675	0.3344329275			

The important constants evaluated at the bifurcation point are

a) A = -0.41 E-2

b) B = 0.77 E-2 (4.5.6)

c) C = 0.20 E-2 .

Because C is non-zero we can use continuation with respect to λ to follow both curves away from the bifurcation point and some results, for the primary curve (4.5.2) and the curve bifurcating away from it, are given in Table 2.

Table 2a : Primary Curve

	$\|\|\underset{\sim}{x}\|\|_2$		$\|\|\underset{\sim}{x}\|\|_2$
$\delta\lambda = -0.05$	0.3093235837	$\delta\lambda = 0.05$	0.3595422714
	0.3094402461		0.3596363202
	0.3094402952		0.3596362852
	0.3094402952		0.3596362852

Table 2b : Bifurcating Curve

	$\|\|\underset{\sim}{x}\|\|_2$		$\|\|\underset{\sim}{x}\|\|_2$
$\delta\lambda = -0.05$	0.5363288624	$\delta\lambda = 0.05$	0.1624069792
	0.5491598965		0.1692256224
	0.5484795767		0.1695624284
	0.5484776727		0.1695631982
	0.5484776727		0.1695631982

CHAPTER 5

IMPERFECT BIFURCATION

5.1 Introduction

To illustrate the ideas of this chapter let us first consider the following simple example of a bifurcation point

$$G(\lambda, x) = (x+\lambda)(x-\lambda) = 0 \qquad x \in R. \tag{5.1.1}$$

Of course $(\lambda^*, x^*) = (0,0)$ is a simple bifurcation point according to definition 4.1. Now suppose that a perturbation, Δ, is introduced so that (5.1.1) becomes

$$G(\lambda, x, \Delta) = (x+\lambda)(x-\lambda) + \Delta = 0. \tag{5.1.2}$$

The qualitative behaviour of the solutions of (5.1.2), for small Δ, is shown in the following diagram.

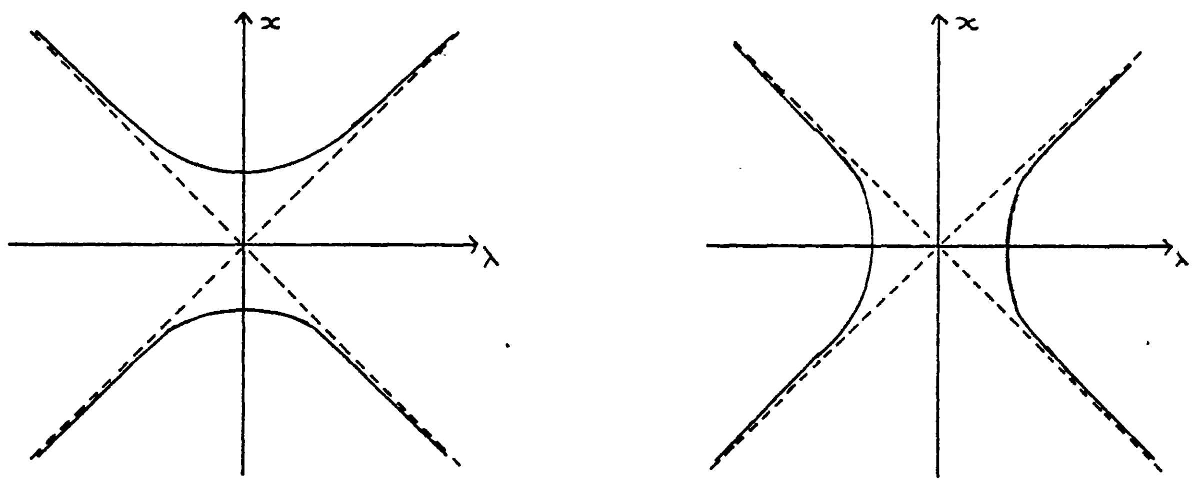

Fig. 5.1a $\Delta < 0$ Fig. 5.1b $\Delta > 0$

Thus it is clear that bifurcation point may disappear under arbitrary small perturbations. The resulting behaviour of the solution curves is typical: the solution branches of the perturbed problem consist of perturbed half-branches of the unperturbed problem, precisely which joins which being determined by the sign of the perturbation. We also note that in Fig. 5.1a the two perturbed solutions cross the x-axis, but in Fig, 5.1b there are no solutions for $\lambda = 0$, but two turning points. This situation is reversed if we consider continuation with respect to x. Finally we also note that, as $\Delta \to 0$, the perturbed solutions tend towards a "corner" at (0,0), making this a singular perturbation problem.

If we are seeking the solution of a continuous problem, e.g. an integral/differential equation, which has a bifurcation point then, in general, the bifurcation point will disappear when the problem is discretized. However, if we are considering bifurcation from the trivial solution, as in chapter 2, this will not happen because the trivial solution will be retained after discretization. Thus our interest in imperfect bifurcation is slightly different to that of Reiss [4]. We also show later that simple turning points, as discussed in chapter 3, are not destroyed by perturbations.

Using the assumptions and notation of sections 4.1 and 4.2, we now consider what happens when the method of section 4.2 is used to determine a simple bifurcation point but, because of the presence of perturbations, there is no exact bifurcation point. Let $C^2(R \times X)$ denote the space of twice-continuously differentiable functions from $R \times X$ to X, which becomes a Banach space under the norm

$$\|G\|_2 = \|G\| + \|G_x\| + \|G_\lambda\| + \|G_{xx}\| + \|G_{\lambda x}\| + \|G_{\lambda\lambda}\| . \tag{5.1.3}$$

Thus for any element $G \in C^2(R \times X)$ we may define the mapping of (4.2.7), and F_G is continuously differentiable. The idea of the proof of the following theorem comes from an observation of Chow, Hale and Mallet-Paret [11,p.162].

Theorem 5.1

If $G_1 \in C^2(R \times X)$ has a simple bifurcation point $y^* = (\lambda^*, x^*, \bar{\psi}^*) \in Y_1$ then $\exists\, \varepsilon_1, \varepsilon_2 > 0$ such that, if $G \in C^2(R \times X)$ and $\|G - G_1\|_2 < \varepsilon_1$, equation (4.2.7)

$$F_G(y) = 0 \qquad (5.1.4)$$

has a solution y(G) which is unique in the ball $\|y - y^*\|_{Y_1} < \varepsilon_2$.
The function y(G): $C^2(R \times X) \longrightarrow Y_1$ is twice-continuously differentiable.

Proof

Define the map e: $C^2(R \times X) \times Y_1 \longrightarrow Y_2$ by

$$e(G, y) = F_G(y). \qquad (5.1.5)$$

Since $e(G_1, y^*) = 0$ and, by Lemma 4.5, $\frac{\partial}{\partial y} e(G_1, y^*)$ is non-singular, the implicit function theorem proves the theorem ∎

The meaning of the above result is that, for sufficiently small perturbations, our method of section 4.2 will still converge to a unique solution of (5.1.4), but will this solution be a bifurcation point of $G(\lambda, x) = 0$? Writing $y(G) = (\lambda(G), x(G), \bar{\psi}(G))$, we may use the perturbation result of Kato [15] to show that $G_x(\lambda(G), x(G))$ and $G_x(\lambda(G), x(G))'$ have one-dimensional null-spaces spanned by continuous functions ϕ(G) and ψ(G) respectively, so that (4.1.1.) a) and b) hold. Also $R\{G_x(\lambda(G), x(G))\}$ will be closed, N will remain a complement of $\{\phi(G)\}$, and M(G) will be continuous. Finally (4.1.1) d) will hold by definition of F and, because $B^2 - 4AC$ (see (4.1.7)) is a continuous function of G, it will remain positive. Thus $(\lambda(G), x(G))$ would be a simple bifurcation point of G except for the fact that there is no necessity for (I-T) G (λ(G), x (G)) to be zero and so, in general,

$$G(\lambda(G);x(G)) = \Delta v \neq 0 \qquad (5.1.6)$$

(v as in (4.2.5)) and $(\lambda(G), x(G))$ is not even a solution of (1.1). However it is clear from the preceding discussion that the function

$$G(\lambda, x) - \Delta v = 0 \qquad (5.1.7)$$

has a simple bifurcation point at $(\lambda(G), x(G))$. This is the key idea of this chapter and we call $(\lambda(G), x(G))$ an imperfect simple bifurcation point of (1.1). We emphasise that although the particular imperfect bifurcation point determined depends on T, the presence or not of a perfect bifurcation point does not.

Equation (5.1.7) motivates us to consider the following problem. Changing notation slightly, let (λ^*, x^*) be a simple bifurcation point of (1.1). For small Δ, what is the form of the solutions of

$$G(\lambda, x) + \Delta v = 0 \qquad (5.1.8)$$

in the neighbourhood of (λ^*, x^*)? This question is answered in section 5.2. In section 5.3 we use Newton's method to determine the solutions and, in section 5.4, the practical problems of dealing with imperfect bifurcation points are discussed. Finally, numerical results are given in section 5.5.

Perturbed or imperfect bifurcation problems have been analysed by several authors, although the dependence of our equations on Δ is particularly simple in comparison. Keener and Keller [18] use an adaption of Keller's iterative method, mentioned in section 2.3; Matkowsky and Reiss [22] and Reiss [4] emphasise that we have a singular perturbation problem and use matched asymptotic expansions; while Chow, Hale and Mallet-Paret [11] and Hale [12] compute functionals which determine the number of solutions in different regions.

Finally we return to our earlier comment that simple turning points are not destroyed by perturbations. This can be proved exactly as in Theorem 5.1, by replacing the function F of (4.2.7) by the F of (3.4.3). If the turning point is not simple then this result does not hold, and the non-simple turning point becomes a cluster of simpler turning points in the same way as a multiple eigen-value of a linear problem becomes a cluster of simpler eigen-values.

5.2 Solutions of the Perturbed Equations

The determination of the solutions of (5.1.8) is reduced to the consideration of a one-dimensional problem, as in the previous chapters, by using the Liapunov-Schmidt method. Thus (4.2.1) becomes

$$w(\delta\lambda,\varepsilon,\Delta) = \delta\lambda w^{*} - \Delta MPv + O((|\delta\lambda|+|\varepsilon|)^2), \qquad (5.2.1)$$

and (4.1.5) can be written

$$f(\delta\lambda,\varepsilon,\Delta) = A\,\delta\lambda^2 + B\,\delta\lambda\varepsilon + C\varepsilon^2 + D\Delta + R_4(\delta\lambda,\varepsilon,\Delta), \qquad (5.2.2)$$

where $D=\psi^{*}(v)$ and R_4 is twice-continuously differentiable with

$$R_4(\delta\lambda,\varepsilon,\Delta) = (|\delta\lambda|+|\varepsilon|)\{O(|\Delta|) + o(|\delta\lambda|+|\varepsilon|)\}. \qquad (5.2.3)$$

We again see the significance of $\psi^{*}(v)$ being non-zero, $D\neq 0$, which allows us to use the implicit function theorem to solve (5.2.2) for Δ in terms of $\delta\lambda$ and ε

$$\Delta(\delta\lambda,\varepsilon) = -(A\,\delta\lambda^2 + B\,\delta\lambda\varepsilon + C\varepsilon^2)/D + o((|\delta\lambda|+|\varepsilon|)^2). \qquad (5.2.4)$$

However we really require the form of the small solutions $\delta\lambda$ and ε for fixed small Δ. Thus we wish to solve

$$\Delta(\delta\lambda,\varepsilon) = \text{const.} \qquad (5.2.5)$$

or

$$\frac{\partial\Delta}{\partial\,\delta\lambda}(\delta\lambda,\varepsilon)\,d\,\delta\lambda + \frac{\partial\Delta}{\partial\varepsilon}(\delta\lambda,\varepsilon)\,d\varepsilon = 0 \qquad (5.2.6)$$

or, to write it in the usual form,

$$\begin{aligned} &\text{a)} \quad \frac{d}{dt}\delta\lambda(t) = P(\delta\lambda,\varepsilon) \\ &\text{b)} \quad \frac{d}{dt}\varepsilon(t) = Q(\delta\lambda,\varepsilon) \end{aligned} \qquad (5.2.7)$$

where t is a parameter, $P(\delta\lambda,\varepsilon) = \frac{\partial\Delta}{\partial\varepsilon}(\delta\lambda,\varepsilon)$ and $Q(\delta\lambda,\varepsilon) = -\frac{\partial\Delta}{\partial\delta\lambda}(\delta\lambda,\varepsilon)$. The formulation (5.2.7) shows that we have a plane autonomous system to analyse and, because this topic is treated extensively in many books on ordinary differential equations (e.g. Coddington and Levinson [7], Hurewicz [13], Friedrichs [10]), we merely sketch the ideas pertinent to our needs.

Both P and Q are zero at the origin and so $(0,0)$ is a singular point of (5.2.7). The behaviour of the solutions near $(0,0)$ is governed by the matrix K

$$\begin{pmatrix} \frac{\partial P}{\partial\delta\lambda}(0,0) & \frac{\partial P}{\partial\varepsilon}(0,0) \\ \frac{\partial Q}{\partial\delta\lambda}(0,0) & \frac{\partial Q}{\partial\varepsilon}(0,0) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2\Delta(0,0)}{\partial\delta\lambda\,\varepsilon} & \frac{\partial^2\Delta(0,0)}{\partial\varepsilon^2} \\ -\frac{\partial^2\Delta(0,0)}{\partial\delta\lambda^2} & -\frac{\partial^2\Delta(0,0)}{\partial\varepsilon\,\delta\lambda} \end{pmatrix} \tag{5.2.8}$$

$$= \frac{1}{D}\begin{pmatrix} -B & -2C \\ 2A & B \end{pmatrix}.$$

The determinant of K, $-(B^2-4AC)/D^2$, is non-zero by (4.1.7), and so $(0,0)$ an isolated singular point of (5.2.7). The eigen-values λ_i of K are real and of opposite sign with

$$\lambda_1 = -\lambda_2 = (B^2-4AC)^{\frac{1}{2}}/D. \tag{5.2.9}$$

Thus the behaviour of the solutions of (5.2.7) near (0,0) is of "saddle-point" type, as depicted in Fig. 5.2.

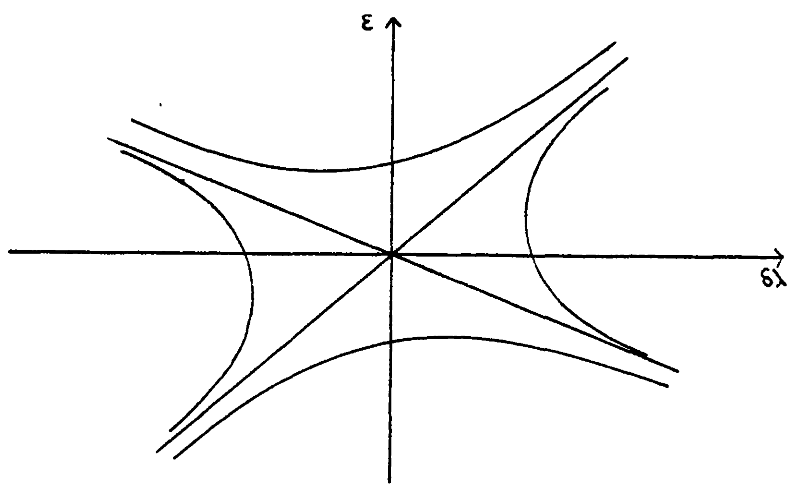

Fig. 5.2 Solution Curve of (5.2.5) for different values of Δ.

The two lines passing through the origin represent the solutions for $\Delta = 0$, i.e. perfect bifurcation considered in the previous chapter. Their slopes at (0,0) are given by the ratios of the components of the eigen-vectors corresponding to λ_i ,

$$\text{a)}\quad (-B-(B^2-4AC)^{1/2})/2C \quad \text{or } 2A/(-B+(B^2-4AC)^{1/2}) \quad \text{for } \lambda_1$$
$$\text{b)}\quad (-B+(B^2-4AC)^{1/2})/2C \quad \text{or } 2A/(-B-(B^2-4AC)^{1/2}) \quad \text{for } \lambda_2. \tag{5.2.10}$$

It can be seen that this agrees with the results of section 4.1.

Having obtained a knowledge of the qualitative behaviour of the solution-curves we can calculate them in terms of $\delta\lambda$ and ε and, as in section 4.1, we consider the three cases $A=0$, $C\neq 0$ and $A=C=0$ separately.

5.2.1 $C\neq 0$

In this case, as shown in Theorem 4.3, the solutions of the unperturbed problem may be developed in terms of $\delta\lambda$,

$$x_i^*(\delta\lambda) = x^* + (p_i\phi^* + w^*)\delta\lambda + o(|\delta\lambda|) \tag{5.2.11}$$

where the p_i are given by (4.1.8). Also $G_x(\lambda^*+\delta\lambda, x_i^*(\delta\lambda))^{-1}$ exists for small non-zero $\delta\lambda$ and

$$\|G_x(\lambda^*+\delta\lambda, x_i^*(\delta\lambda))^{-1}\| = O(|\delta\lambda|^{-1}). \tag{5.2.12}$$

Thus we may use the implicit function theorem to develop perturbed solutions in Δ about $(\lambda^*+\delta\lambda, x_i^*(\delta\lambda))$ but as $|\delta\lambda|\to 0$ the region of existence of these perturbed solutions also shrinks to zero (in fact Δ must be $O(|\delta\lambda|^2)$). Thus, for any fixed Δ, we cannot obtain solutions near $|\delta\lambda| = 0$. The behaviour of the solutions here is governed by $\operatorname{sgn}(\Delta C D)$, as can be seen by solving the pair of equations

$$\text{a)}\quad f(\delta\lambda, \varepsilon, \Delta) = 0$$
$$\text{b)}\quad \frac{\partial f}{\partial \varepsilon}(\delta\lambda, \varepsilon, \Delta) = 0 \tag{5.2.13}$$

which determine the singular solutions of (5.2.2) when we are parametrising ε by $\delta\lambda$ and Δ. (5.2.13b) can be written

$$2C\varepsilon + B\delta\lambda + \frac{\partial}{\partial\varepsilon}R_4(\delta\lambda,\varepsilon,\Delta) = 0, \tag{5.2.14}$$

and, as $C \neq 0$, the implicit function theorem gives a unique solution $\varepsilon_E(\delta\lambda,\Delta)$ of the form

$$\varepsilon_E(\delta\lambda,\Delta) = -B\delta\lambda/2C + O(|\Delta|) + o(|\delta\lambda|). \tag{5.2.15}$$

Inserting this expression into (5.2.13a) gives

$$\Delta D - (B^2-4AC)\delta\lambda^2/4C + o(|\delta\lambda|^2+|\Delta|) = 0 \tag{5.2.16}$$

and, as $B^2-4AC>0$ by (4.1.7), the significance of $\mathrm{sgn}(\Delta C D)$ is clear. If $\Delta CD<0$ then (5.2.16) has no small solutions, while if $\Delta CD>0$ there are two solutions

$$\delta\lambda^{\pm}(\Delta) = \pm\{4\Delta CD/(B^2-4AC)\}^{\frac{1}{2}} + o(|\Delta|^{\frac{1}{2}}). \tag{5.2.17}$$

Thus if $\Delta CD<0$ the solution curves can be extended across the x-axis and (5.2.2) has two solutions for $\delta\lambda=0$

$$\varepsilon_i(\Delta) = (-1)^i(-\Delta D/C)^{\frac{1}{2}} + o(|\Delta|^{\frac{1}{2}}). \tag{5.2.18}$$

Using (5.2.1) this extends to the solutions

$$\begin{aligned} x_i(\Delta) &= x^* + \varepsilon_i(\Delta)\phi^* + w(0,\varepsilon_i(\Delta),\Delta) \\ &= x^* + (-1)^i(-\Delta D/C)^{\frac{1}{2}}\phi^* + o(|\Delta|^{\frac{1}{2}}) \end{aligned} \tag{5.2.19}$$

of (5.1.8). However, if $\Delta CD>0$, Chow, Hale and Mallet-Paret [11, § 8] have shown that, for sufficiently small $|\Delta|$ no small solutions of (5.2.2) exist for $\delta\lambda^-(\Delta)<\delta\lambda<\delta\lambda^+(\Delta)$, $\delta\lambda^{\pm}(\Delta)$ given by (5.2.17). This follows from the fact that

$$f(\delta\lambda,\varepsilon_E(\delta\lambda,\Delta),\Delta) = \Delta D - (B^2-4AC)\delta\lambda^2/4C + o(|\delta\lambda|^2+|\Delta|) \tag{5.2.20}$$

and so is the same sign as C; but

$$\frac{\partial^2 f}{\partial\varepsilon^2}(\delta\lambda,\varepsilon,\Delta) = 2C + \frac{\partial^2 R_4}{\partial\varepsilon^2}(\delta\lambda,\varepsilon,\Delta) \tag{5.2.21}$$

shows that $f(\delta\lambda,\varepsilon_E(\delta\lambda,\Delta),\Delta)$ is a maximum, if $C<0$, or a minimum, if $C>0$, for f over ε with fixed $\delta\lambda$ and Δ. Hence no solutions of (5.2.2) will exist between the $\delta\lambda$-values given by (5.2.17).

For fixed Δ it is easy to see that $(\delta\lambda^{\pm}(\Delta), \varepsilon_E(\delta\lambda^{\pm}(\Delta),\Delta))$ are simple turning points with respect to $\delta\lambda$ of (5.2.2), because

$$\frac{\partial f}{\partial \delta\lambda}(\delta\lambda^{\pm}(\Delta), \varepsilon_E(\delta\lambda^{\pm}(\Delta),\Delta)) = \mp\{(B^2-4AC)\Delta D/C\}^{\frac{1}{2}} + o(|\Delta|^{\frac{1}{2}}) \qquad (5.2.22)$$

means that condition (3.1.1d) is satisfied and (5.2.21) shows that (3.3.8) holds. This implies that $(\lambda^*(\Delta), x^*(\Delta))$ where

$$\begin{aligned} &\text{a)} \quad \lambda^*(\Delta) = \lambda^* + \delta\lambda^{\pm}(\Delta) \\ &\text{b)} \quad x^*(\Delta) = x^* + \varepsilon_E(\delta\lambda^{\pm}(\Delta),\Delta)\phi^* + w(\delta\lambda^{\pm}(\Delta), \varepsilon_E(\delta\lambda^{\pm}(\Delta),\Delta),\Delta) \end{aligned} \qquad (5.2.23)$$

are simple turning points of (5.1.8), which can also be proved by straight-forward verification. For, by definition,

$$\phi^* + \frac{\partial w}{\partial \varepsilon}(\delta\lambda^{\pm}(\Delta), \varepsilon_E(\delta\lambda^{\pm}(\Delta),\Delta),\Delta) \qquad (5.2.24)$$

is in the null-space of $G_x(\lambda^*(\Delta), x^*(\Delta))$ and so, from the perturbation results of Kato, (3.1.1) a), b) and c) hold. Additionally $\mathcal{N}\{G_x(\lambda^*(\Delta), x^*(\Delta))'\}$ contains an element of the form

$$\begin{aligned} \psi^*(\Delta) &= \psi^* + \psi_1^*(\Delta) \qquad\qquad (I-P_1)\psi^*(\Delta) = 0 \\ &= \psi^* \mp \{4CD\Delta/(B^2-4AC)\}^{\frac{1}{2}} M_1\{G_{x\lambda}(\lambda^*, x^*)'\psi^* \\ &\quad -\frac{B}{2C}(G_{xx}(\lambda^*, x^*)\phi^*)'\psi^* + (G_{xx}(\lambda^*, x^*)w^*)'\psi^*\} + o(|\Delta|^{1/2}) \end{aligned} \qquad (5.2.25)$$

and thus

$$\psi^*(\Delta)(G_\lambda(\lambda^*(\Delta), x^*(\Delta))) = \mp\{(B^2-4AC)\Delta D/C\}^{1/2} + o(|\Delta|^{1/2}) \qquad (5.2.26)$$

and (3.1.1d) holds. Finally (3.3.8) follows immediately from $C \neq 0$.

5.2.2. $A \neq 0$

In this case a similar analysis to section (5.2.1) can be carried out. By Theorem 4.3 the unperturbed solutions can be developed in terms of ε

$$\begin{aligned} &\text{a)} \quad \lambda_i^*(\varepsilon) = \lambda^* + q_i\varepsilon + o(|\varepsilon|) \\ &\text{b)} \quad x_i^*(\varepsilon) = x^* + \varepsilon(\phi^* + q_i w^*) + o(|\varepsilon|) \end{aligned} \qquad (5.2.27)$$

where the q_i are given by (4.1.18), and $G_x(\lambda_i^*(\varepsilon), x_i^*(\varepsilon)) + G_\lambda(\lambda_i^*(\varepsilon), x_i^*(\varepsilon))$ regarded as an operator from R x N to X, is non-singular for small non-zero ε , with

$$\|\{G_x(\lambda_i^*(\varepsilon),x_i^*(\varepsilon))+G_\lambda(\lambda_i^*(\varepsilon),x_i^*(\varepsilon))\}^{-1}\| = O(|\varepsilon|^{-1}). \tag{5.2.28}$$

Perturbed solutions in Δ may be developed about $(\lambda_i^*(\varepsilon),x_i^*(\varepsilon))$, but only for $\Delta = O(|\varepsilon|^2)$. Near $\varepsilon=0$ the form of the solutions depends on the singular points given by

$$\begin{aligned} &a) \quad f(\delta\lambda,\varepsilon,\Delta) = 0 \\ &b) \quad \frac{\partial f}{\partial\delta\lambda}(\delta\lambda,\varepsilon,\Delta) = 0 . \end{aligned} \tag{5.2.29}$$

Solving (5.2.29b) gives

$$\delta\lambda_\varepsilon(\varepsilon,\Delta) = -B\varepsilon/2A + O(|\Delta|) + o(|\varepsilon|) \tag{5.2.30}$$

and, inserting this into (5.2.29a), we obtain

$$\Delta D-(B^2-4AC)\varepsilon^2/4A + o(|\varepsilon|^2+|\Delta|) = 0. \tag{5.2.31}$$

If $\Delta AD<0$, (5.2.31) has no small solutions, while if $\Delta AD>0$ there are two solutions

$$\varepsilon^\pm(\Delta) = \pm\{4\Delta AD/(B^2-4AC)\}^{1/2} + o(|\Delta|^{1/2}). \tag{5.2.32}$$

Thus, for $\Delta AD<0$, (5.2.2) has two solutions for $\varepsilon=0$, namely

$$\delta\lambda_i(\Delta) = (-1)^i(-\Delta D/A)^{1/2} + o(|\Delta|^{1/2}) \tag{5.2.33}$$

which extend to solutions

$$\begin{aligned} &a) \quad \lambda_i(\Delta) = \lambda^* + \delta\lambda_i(\Delta) \\ &b) \quad x_i(\Delta) = x^* + w(\delta\lambda_i(\Delta),0,\Delta). \end{aligned} \tag{5.2.34}$$

However for $\Delta AD>0$ there are no solutions for $\varepsilon^-(\Delta)<\varepsilon<\varepsilon^+(\Delta)$, and $(\delta\lambda_\varepsilon(\varepsilon^\pm(\Delta),\Delta),\varepsilon^\pm(\Delta))$ are simple turning points with respect to ε of (5.2.2). This implies that $(\lambda^*(\Delta),x^*(\Delta))$, given by

$$\begin{aligned} &a) \quad \lambda^*(\Delta) = \lambda^* + \delta\lambda_\varepsilon(\varepsilon^\pm(\Delta),\Delta) \\ &b) \quad x^*(\Delta) = x^* + \varepsilon^\pm(\Delta)\phi^* + w(\delta\lambda_\varepsilon(\varepsilon^\pm(\Delta),\Delta),\varepsilon^\pm(\Delta),\Delta); \end{aligned} \tag{5.2.35}$$

are simple turning points of (5.1.8) with respect to ε. However we must remember that this means they are simple turning points of the operator H: $R\times(R \times N) \longrightarrow X$ given by

$$H(\varepsilon,y) = G(\lambda, x^*+\varepsilon\phi^*+w) \tag{5.2.36}$$

where $y=(\lambda,w)$. Then, by definition,

$$y = \left(1, \frac{\partial w}{\partial\delta\lambda}(\delta\lambda_\varepsilon(\varepsilon^\pm(\Delta),\Delta),\varepsilon^\pm(\Delta),\Delta)\right) \tag{5.2.37}$$

is an element of the null-space of $G_x(\lambda^*(\Delta),x^*(\Delta))+G_\lambda(\lambda^*(\Delta),x^*(\Delta))$ and conditions (3.1.1) quickly follow from perturbation results.

5.2.3 A = C = 0

In this case it follows from Theorem 4.4 that one of the unperturbed solutions can be parametrised by $\delta\lambda$, and written

$$x_1^*(\delta\lambda) = x^* + \delta\lambda\, w^* + o(|\delta\lambda|), \tag{5.2.38}$$

and the other by ϵ, and written

$$\begin{aligned} &\text{a)} \quad \lambda_2^*(\epsilon) = \lambda^* + o(|\epsilon|) \\ &\text{b)} \quad x_2^*(\epsilon) = x^* + \epsilon\phi^* + o(|\epsilon|). \end{aligned} \tag{5.2.39}$$

Perturbed solutions may be developed about $x_1^*(\delta\lambda)$ for $\Delta = O(|\delta\lambda|^2)$ and about $(\lambda_2^*(\epsilon), x_2^*(\epsilon))$ for $\Delta = O(|\epsilon|^2)$. Near (0,0) the behaviour of the solutions depends on $\operatorname{sgn}(\Delta BD)$. If this is negative then two small solutions of (5.2.2) with $\epsilon = \delta\lambda$ exist, namely

$$\epsilon_i(\Delta) = \delta\lambda_i(\Delta) = (-1)^i(-D\Delta/B)^{1/2} + o(|\Delta|^{1/2}) \tag{5.2.40}$$

and these extend to solutions

$$\begin{aligned} &\text{a)} \quad \lambda_i(\Delta) = \lambda^* + \delta\lambda_i(\Delta) \\ &\text{b)} \quad x_i(\Delta) = x^* + \epsilon_i(\Delta)\phi^* + w(\delta\lambda_i(\Delta), \epsilon_i(\Delta), \Delta). \end{aligned} \tag{5.2.41}$$

If ΔBD is positive then two small solutions with $\epsilon = -\delta\lambda$ exist

$$\epsilon_i(\Delta) = -\delta\lambda_i(\Delta) = (-1)^i(D\Delta/B)^{1/2} + o(|\Delta|^{1/2}) \tag{5.2.42}$$

and these extend in the same way as (5.2.41).

5.3 Newton's Method at Imperfect Bifurcation Points

In this section we assume that we already have an imperfect bifurcation point (λ^*, x^*) and now we wish to move onto the near-by solution curves. As usual we apply Corollary 2.4 with starting values obtained from section 5.2.

First let us consider the case $C \neq 0$. If both $|\delta\lambda|$ and $|\Delta/\delta\lambda^2|$ are sufficiently small then

$$p_i(\delta\lambda,\Delta) = \left(-B + (-1)^i\{B^2 - 4C(A + D\Delta/\delta\lambda^2)\}^{1/2}\right)/2C \qquad i = 1,2 \tag{5.3.1}$$

exist and we can use this starting value instead of $p_i(\delta\lambda)$ in (4.4.6).

Theorem 5.2.

There exists $K>0$ such that for fixed $\delta\lambda$, $|\delta\lambda|$ sufficiently small, and fixed Δ, $|\Delta| < K|\delta\lambda^2|^2$, the Newton iteration

$$x^{n+1}(\delta\lambda) = x^n(\delta\lambda) - G_x(\lambda^* + \delta\lambda, x^n(\delta\lambda))^{-1}\{G(\lambda^* + \delta\lambda, x^n(\delta\lambda)) + \Delta v\}, \tag{5.3.2}$$

with starting approximation

$$x^0(\delta\lambda) = x^* + \delta\lambda\left(p_i(\delta\lambda,\Delta)\phi^* + w^*\right), \tag{5.3.3}$$

converges to the solution-curves of (5.1.8).

Proof

We define $T(|\delta\lambda|, x) = G(\lambda^* + \delta\lambda, x) + \Delta v$ and verify the conditions of Corollary 2.4.

$$\begin{aligned} T_x(|\delta\lambda|, x) = G_x(\lambda^*, x^*) + \delta\lambda\{ & G_{x\lambda}(\lambda^*, x^*) \\ & + G_{xx}(\lambda^*, x^*)(p_i(\delta\lambda,\Delta)\phi^* + w^*) + L_1(\delta\lambda,\Delta)\} \end{aligned} \tag{5.3.4}$$

where $L_1(\delta\lambda,\Delta) \to 0$ as $\delta\lambda \to 0$ with $|\Delta| < K|\delta\lambda|^2$. As

$$\begin{aligned} \psi^*\left(G_{x\lambda}(\lambda^*, x^*)\phi^* + G_{xx}(\lambda^*, x^*)(p_i(\delta\lambda,\Delta)\phi^* + w^*)\phi^*\right. \\ = 2p_i(\delta\lambda,\Delta)C + B \end{aligned} \tag{5.3.5}$$

is non-zero from (5.3.1), we may use Lemma 2.1 and the Banach perturbation theorem to show that $T_x(|\delta\lambda|, x^0(\delta\lambda))^{-1}$ exists for sufficiently small non-zero $\delta\lambda$ and

$$\text{a)} \quad \|T_x(|\delta\lambda|, x^0(\delta\lambda))^{-1}\| = O(|\delta\lambda|^{-1}),$$

$$\text{but} \quad \text{b)} \quad \|T_x(|\delta\lambda|, x^0(\delta\lambda))^{-1}\| = O(1) \tag{5.3.6}$$

when restricted to $\mathcal{R}\{G_x(\lambda^*, x^*)\}$.

$$T(|\delta\lambda|, x^0(\delta\lambda)) = G(\lambda^* + \delta\lambda, x^0(\delta\lambda)) + \Delta v \tag{5.3.7}$$

$$= \frac{\delta\lambda^2}{2}\{G_{xx}(\lambda^*,x^*)(p_i(\delta\lambda,\Delta)\phi^*+w^*)^2+G_{\lambda\lambda}(\lambda^*,x^*)$$
$$+2G_{x\lambda}(\lambda^*,x^*)(p_i(\delta\lambda,\Delta)\phi^*+w^*)+\frac{2\Delta}{\delta\lambda^2}v\}+o(|\delta\lambda|^2)$$

and so $\|T(|\delta\lambda|, x^\circ(\delta\lambda))\| = O(|\delta\lambda|^2)$, (5.3.8)

but by definition of $p_i(\delta\lambda,\Delta)$

$$|\psi^*(T(|\delta\lambda|, x^\circ(\delta\lambda)))| = o(|\delta\lambda|^2). \tag{5.3.9}$$

Combining (5.3.6), (5.3.8) and (5.3.9) gives

$$\|T_x(|\delta\lambda|, x^\circ(\delta\lambda))^{-1}T(|\delta\lambda|, x^\circ(\delta\lambda))\| \leq \eta(\delta\lambda) = o(|\delta\lambda|). \tag{5.3.10}$$

As in Theorem 4.7 for $\|z_i - x^\circ(\delta\lambda)\| < 2\eta(\delta\lambda)$ $i=1,2$

$$\|T_x(|\delta\lambda|, x^\circ(\delta\lambda))^{-1}\{T_x(|\delta\lambda|,z_1)-T_x(|\delta\lambda|,z_2)\}\| \leq L(\delta\lambda)\|z_1-z_2\| \cdot \tag{5.3.11}$$

where $L(\delta\lambda) = O(|\delta\lambda|^{-1})$. Thus

$$\eta(\delta\lambda)L(\delta\lambda) = o(1) \tag{5.3.12}$$

and the theorem is proved ∎

Now we consider what to do when $|\Delta|$ is not sufficiently small for Theorem 5.2 to hold. In this case we simply determine particular points on the solution curves, according to the sign of $\Delta C D$.

If $\Delta C D < 0$ we know that the solution curves cross the x-axis and so $\delta\lambda$ is set to zero and we compute (5.2.19).

<u>Theorem 5.3</u>

For fixed Δ, $|\Delta|$ sufficiently small and $\Delta C D < 0$, the Newton iteration

$$x^{n+1}(\Delta) = x^n(\Delta) - G_x(\lambda^*, x^n(\Delta))^{-1}\{G(\lambda^*, x^n(\Delta)) + \Delta v\}, \tag{5.3.13}$$

with starting approximation

$$x^\circ(\Delta) = x^* \pm (-\Delta D/C)^{1/2}\phi^*, \tag{5.3.14}$$

will converge to the solutions (5.2.19).

Proof

As usual we verify the conditions of Corollary 2.4 with

$$T(|\Delta|, x) = G(\lambda^*, x) + \Delta v \tag{5.3.15}$$

and, because $C \neq 0$, Lemma 2.1 and the Banach perturbation theorem give

$$\|T_x(|\Delta|, x^\circ(\Delta))^{-1}\| = O(|\Delta|^{-1/2}) \tag{5.3.16}$$

but, if restricted to $\mathcal{R}\{G_x(\lambda^*, x^*)\}$,

$$\|T_x(|\Delta|, x^\circ(\Delta))^{-1}\| = O(1). \tag{5.3.17}$$

$$T(|\Delta|, x^\circ(\Delta)) = (-\Delta D/C)\, G_{xx}(\lambda^*, x^*)\phi^*\phi^* + \Delta v + o(|\Delta|) \tag{5.3.18}$$

and so

$$\|T(|\Delta|, x^\circ(\Delta))\| = O(|\Delta|) \tag{5.3.19}$$

but $$|\psi^*(T(|\Delta|, x^\circ(\Delta)))| = o(|\Delta|). \tag{5.3.20}$$

Combining (5.3.16), (5.3.17), (5.3.19) and (5.3.20), we obtain

$$\|T_x(|\Delta|, x^\circ(\Delta))^{-1} T(|\Delta|, x^\circ(\Delta))\| \leq \eta(\Delta) = o(|\Delta|^{1/2}). \tag{5.3.21}$$

As in previous theorems

$$\|T_x(|\Delta|, x^\circ(\Delta))^{-1}\{T_x(|\Delta|, z_1) - T_x(|\Delta|, z_2)\}\| \leq L(|\Delta|)\|z_1 - z_2\| \tag{5.3.22}$$

for $\|z_i - x^\circ(\Delta)\| < 2\eta(\Delta)$ $i = 1, 2$ with $L(|\Delta|) = O(|\Delta|^{-1/2})$ and so

$$\eta(\Delta) L(\Delta) = o(1) \blacksquare$$

If $\Delta C D > 0$ then each solution curve has a simple turning point (5.2.23), and it is these that we compute. Naturally we use the method of section 3.4 for this, with $F(\Delta, y) : R \times Y \to Y$ defined by (3.4.3) with $G(\lambda, x) + \Delta v = 0$ in (3.4.1a). The obvious choice for ℓ in (3.4.1c) is ℓ^* of (4.2.1). In the following theorem we freely use the notation of section 3.4 .

Theorem 5.4

If G_{xx} and $G_{x\lambda}$ possess uniform Lipschitz constants in a neighbourhood of (λ^*, x^*) then for fixed Δ, $|\Delta|$ sufficiently small and $\Delta CD > 0$, the Newton iteration

$$y^{n+1}(\Delta) = y^n(\Delta) - F_y(\Delta, y^n(\Delta))^{-1} F(\Delta, y^n(\Delta)); \tag{5.3.23}$$

with starting approximation $y^0(\Delta) = (\lambda^0(\Delta), x^0(\Delta), \bar{\phi}^0(\Delta))$ given by

$$\begin{aligned} &a) \quad \lambda^0(\Delta) = \lambda^* + \delta\lambda(\Delta) = \lambda^* \pm \{4\Delta CD/(B^2 - 4AC)\}^{1/2} \\ &b) \quad x^0(\Delta) = x^* + (w^* - B\phi^*/2C)\,\delta\lambda(\Delta) \end{aligned} \tag{5.3.24}$$

and $\bar{\phi}^0(\Delta)$ an element of the null-space of the smallest eigen-value of $G_x(\lambda^0(\Delta), x^0(\Delta))$, normalised so that $l^*(\bar{\phi}^0(\Delta)) = 1$; will converge quadratically to the simple turning points of (5.1.8) given by (5.2.23).

Proof

We apply Corollary 2.4 with $T(|\Delta|, y) = F(\Delta, y)$. The elements spanning the null-spaces of $F_y(0, y^*)$ and $F_y(0, y^*)'$, with $y^* = (\lambda^*, x^*, \phi^*)$, are easily calculated to be $\phi^*_y = (\lambda, x, \phi)$ where

$$\begin{aligned} &a) \quad \lambda = 1 \\ &b) \quad x = w^* - B\phi^*/2C \\ &c) \quad \phi = l^*\{MP(G_{x\lambda}(\lambda^*, x^*)\phi^* + G_{xx}(\lambda^*, x^*)\phi^*(w^* - B\phi^*/2C)\}\,\phi^* \\ &\qquad\quad - MP\{G_{x\lambda}(\lambda^*, x^*)\phi^* + G_{xx}(\lambda^*, x^*)\phi^*(w^* - B\phi^*/2C)\} \end{aligned} \tag{5.3.25}$$

and $\psi^*_y = (\psi^*, 0, 0)$ respectively.

Looking at $T_y(|\Delta|, y^0(\Delta))$ we have

$$\begin{aligned} a) \quad & G_x(\lambda^0(\Delta), x^0(\Delta)) + G_\lambda(\lambda^0(\Delta), x^0(\Delta)) = G_x(\lambda^*, x^*) + G_\lambda(\lambda^*, x^*) \\ & + \delta\lambda(\Delta)\{2G_{x\lambda}(\lambda^*, x^*) + G_{xx}(\lambda^*, x^*)(w^* - B\phi^*/2C) + G_{\lambda\lambda}(\lambda^*, x^*)\} \\ & + o(|\Delta|^{1/2}) \end{aligned} \tag{5.3.26}$$

b) $G_x(\lambda^\circ(\Delta), x^\circ(\Delta)) + G_{xx}(\lambda^\circ(\Delta), x^\circ(\Delta))\phi^\circ(\Delta) + G_{x\lambda}(\lambda^\circ(\Delta), x^\circ(\Delta))\phi^\circ(\Delta)$
$= G_x(\lambda^*, x^*) + G_{xx}(\lambda^*, x^*)\phi^* + G_{x\lambda}(\lambda^*, x^*)\phi^* + O(|\Delta|^{1/2})$.

Because

$$\psi^*(2G_{x\lambda}(\lambda^*, x^*)(w^* - B\phi^*/2c) + G_{xx}(\lambda^*, x^*)(w^* - B\phi^*/2c)^2 + G_{\lambda\lambda}(\lambda^*, x^*)) = -(B^2 - 4Ac)/2c \neq 0 \qquad (5.3.27)$$

we can apply Lemma 2.1 and the Banach perturbation theorem to show that $T_y(|\Delta|, y^\circ(\Delta))^{-1}$ exists, for sufficiently small non-zero $|\Delta|$, and

$$\| T_y(|\Delta|, y^\circ(\Delta))^{-1} \| = O(|\Delta|^{-1/2}) \qquad (5.3.28)$$

but, when restricted to $y = (x, z, \mu)$, such that $x \in R\{G_x(\lambda^*, x^*)\}$

$$\| T_y(|\Delta|, y^\circ(\Delta))^{-1} \| = O(1). \qquad (5.3.29)$$

Now we look at $T(|\Delta|, y^\circ(\Delta))$ which can be written

a) $G(\lambda^\circ(\Delta), x^\circ(\Delta)) + \Delta v = \{G_\lambda(\lambda^*, x^*) + G_x(\lambda^*, x^*)(w^* - B\phi^*/2c)\}\delta\lambda(\Delta)$
$+ \Delta v + \frac{\delta\lambda(\Delta)^2}{2}\{G_{\lambda\lambda}(\lambda^*, x^*) + 2G_{x\lambda}(\lambda^*, x^*)(w^* - B\phi^*/2c) + G_{xx}(\lambda^*, x^*)(w^* - B\phi^*/2c)^2\} + o(|\Delta|)$

b) $G_x(\lambda^\circ(\Delta), x^\circ(\Delta))\bar{\phi}^\circ(\Delta) = 0$ (5.3.30)

c) $L^*(\bar{\phi}^\circ(\Delta)) - 1 = 0$.

Because

$$\Delta D + \tfrac{1}{2}\psi^*\{G_{\lambda\lambda}(\lambda^*, x^*) + 2G_{x\lambda}(\lambda^*, x^*)(w^* - B\phi^*/2c) + G_{xx}(\lambda^*, x^*)(w^* - B\phi^*/2c)^2\} = \Delta D - \delta\lambda(\Delta)^2(B^2 - 4Ac)/4c = 0, \qquad (5.3.31)$$

by definition of $\delta\lambda(\Delta)$,

$$\|T(|\Delta|, y^\circ(\Delta))\| = O(|\Delta|) \qquad (5.3.32)$$

but

$$|\psi_y^*(T(|\Delta|, y^\circ(\Delta)))| = o(|\Delta|). \qquad (5.3.33)$$

Thus combining (5.3.28), (5.3.29), (5.3.32) and (5.3.33), we have

$$\| T_y(|\Delta|, y^\circ(\Delta))^{-1} T(|\Delta|, y^\circ(\Delta)) \| \leq \eta(\Delta) = o(|\Delta|^{1/2}). \qquad (5.3.34)$$

The smoothness of G gives the usual Lipschitz condition

$$\| T_y(|\Delta|, y^\circ(\Delta))^{-1}\{T_y(|\Delta|, y_1) - T_y(|\Delta|, y_2)\} \| \leq L(\Delta)\|y_1 - y_2\| \qquad (5.3.35)$$

for $\|y_i - y^\circ(\Delta)\| < 2\eta(\Delta)$ $i = 1, 2$, with $L(\Delta) = O(|\Delta|^{-1/2})$. Thus $\eta(\Delta)L(\Delta) = o(1)$ and the theorem is proved ∎

For $A \neq 0$ very similar theorems can be proved about the function H: $R \times (R \times N) \longrightarrow X$, defined by

$$H(\varepsilon,(\lambda,w)) = G(\lambda, x^* + \varepsilon\phi^* + w) \qquad (5.3.36)$$

and so we just state them without proof. The analogue to (5.3.1) is

$$q_i(\varepsilon,\Delta) = \{-B + (-1)^i (B^2 - 4A(C + \Delta D/\varepsilon^2))^{1/2}\}/2A \qquad (5.3.37)$$

and the following is the analogue to theorem 5.2.

Theorem 5.5

There exists $K>0$ such that for fixed ε, $|\varepsilon|$ sufficiently small, and fixed Δ, $|\Delta| < K|\varepsilon|^2$, the Newton iteration

$$z^{n+1}(\varepsilon) = z^n(\varepsilon) - H_z(\varepsilon, z^n(\varepsilon))^{-1}\{H(\varepsilon, z^n(\varepsilon)) + \Delta v\}, \qquad (5.3.38)$$

with starting approximation

$$z^0(\varepsilon) = (\lambda^* + q_i(\varepsilon,\Delta)\varepsilon, q_i(\varepsilon,\Delta)\varepsilon w^*), \qquad (5.3.39)$$

will converge to the solution curves of (5.1.8).

For larger $|\Delta|$ the point on the solution curves, which we shall determine, depends on $\mathrm{sgn}(\Delta A D)$. For $\Delta A D < 0$ there are solutions with $\varepsilon = 0$ given by (5.2.34), and the following theorem computes them.

Theorem 5.6

For fixed Δ, $|\Delta|$ sufficiently small and $\Delta A D < 0$, the Newton iteration

$$z^{n+1}(\Delta) = z^n(\Delta) - H_z(0, z^n(\Delta))^{-1}\{H(0, z^n(\Delta)) + \Delta v\}, \qquad (5.3.40)$$

with starting approximation

$$z^0(\Delta) = (\lambda^* \pm (-\Delta D/A)^{1/2}, \pm(-\Delta D/A)^{1/2} w^*), \qquad (5.3.41)$$

will converge to the solutions given by (5.2.34).

For $\Delta AD > 0$ we shall determine the two simple turning points of $H(\varepsilon, z) = 0$ given by (5.2.35). Thus, in the following theorem, F of (3.4.3) uses $H(\varepsilon, z) + \Delta v$ in its definition (3.4.1), instead of G.

Theorem 5.7

If H_{zz} and $H_{z\varepsilon}$ satisfy a uniform Lipschitz condition in a neighbourhood of $(\varepsilon, z) = (0, (\lambda^*, 0))$ then for fixed Δ, $|\Delta|$ sufficiently small and $\Delta AD > 0$, the Newton iteration

$$y^{n+1}(\Delta) = y^n(\Delta) - F_y(\Delta, y^n(\Delta))^{-1} F(\Delta, y^n(\Delta)) ; \qquad (5.3.42)$$

with starting approximation $y^0(\Delta) = (\lambda^0(\Delta), x^0(\Delta), \bar{\phi}^0(\Delta))$ where

$$\begin{aligned} &\text{a)} \quad \varepsilon^0(\Delta) = \pm \{4\Delta AD/(B^2 - 4AC)\}^{1/2} \\ &\text{b)} \quad z^0(\Delta) = (\lambda, w) \\ &\qquad = (\lambda^* - B\varepsilon^0(\Delta)/2A, -B\varepsilon^0(\Delta)w^*/2A) \end{aligned} \qquad (5.3.43)$$

and $\bar{\phi}^0(\Delta)$ is the element of the null-space of the smallest eigenvalue of $H_z(\varepsilon(\Delta), (\lambda^0(\Delta), \lambda^0(\Delta)w^*))$ normalised so that its λ-component is unity; will converge to the two simple turning points given by (5.2.35).

For A = C = 0, and $|\Delta|$ sufficiently small in relation to $|\delta\lambda|^2$ and $|\varepsilon|^2$, we state a theorem similar to Theorems 5.2 and 5.5. Now, however, both parametrisation by $\delta\lambda$ and ε must be used.

Theorem 5.8

There exists $K > 0$ such that for fixed $\delta\lambda$, $|\delta\lambda|$ sufficiently small, and fixed Δ, $|\Delta| < K|\varepsilon|^2$, the Newton iteration

$$x^{n+1}(\delta\lambda) = x^n(\delta\lambda) - G_x(\lambda^* + \delta\lambda, x^n(\delta\lambda))^{-1}\{G(\lambda^* + \delta\lambda, x^n(\delta\lambda)) + \Delta v\}, \qquad (5.3.44)$$

with starting approximation

$$x^0(\delta\lambda) = x^* - \Delta D\phi^*/B\delta\lambda + \delta\lambda w^* , \qquad (5.3.45)$$

converges to the solutions of (5.1.8) which are perturbations of (5.2.38); while for fixed ε, $|\varepsilon|$ sufficiently small, and fixed Δ, $|\Delta| < K|\varepsilon|^2$, the Newton iteration

$$z^{n+1}(\varepsilon) = z^n(\varepsilon) - H_z(\varepsilon, z^n(\varepsilon))^{-1}\{H(\varepsilon, z^n(\varepsilon)) + \Delta v\}, \tag{5.3.46}$$

with starting approximation

$$z^0(\varepsilon) = (\lambda^* - \Delta D/B\varepsilon, -\Delta D w^*/B\varepsilon), \tag{5.3.47}$$

converges to the solutions of (5.1.8) which are perturbations of (5.2.39).

If $|\Delta|$ is not sufficiently small to allow us to use Theorem 5.8, we may compute points near to the solutions given by (5.2.41). Of course we cannot determine these exact points but what is important is getting onto the appropriate solution curve.

Theorem 5.9

For fixed Δ, $|\Delta|$ sufficiently small, the Newton iteration

$$x^{n+1}(\Delta) = x^n(\Delta) - G_x(\lambda^* \pm |\Delta D/B|^{1/2}, x^n(\Delta))^{-1}\{G(\lambda^* \pm |\Delta D/B|^{1/2}, x^n(\Delta)) + \Delta v\}, \tag{5.3.48}$$

with starting approximation

$$x^0(\Delta) = x^* \mp \operatorname{sgn}(\Delta B D)|\Delta D/B|^{1/2}\phi^* \pm |\Delta D/B|^{1/2} w^*, \tag{5.3.49}$$

converges to a point on the solution curves near (5.2.41).

Finally we mention that the ideas of this section can also be applied to Keller's formulation, briefly reviewed at the end of section 4.4.

5.4 Implementation in R^n

In this section we discuss the procedure to be followed when dealing with an imperfect bifurcation point and the practical problems of moving onto a nearby solution-curve.

As was shown in section 5.1, we use the method of section 4.2 to compute a simple bifurcation point, and only afterwards decide whether it is imperfect or not. Thus, following the implementation details of section 4.3, we choose $\underset{\sim}{v} = \underset{\sim}{e}_t$ and arrive at

$$\begin{aligned}
&\text{a)}\quad G(\lambda^*, \underset{\sim}{x}^*) = \Delta \underset{\sim}{e}_t \\
&\text{b)}\quad G_x(\lambda^*, \underset{\sim}{x}^*)\underset{\sim}{\phi}^* = 0 \qquad \underset{\sim}{e}_s^T \underset{\sim}{\phi}^* = 1 \qquad (5.4.1)\\
&\text{c)}\quad G_x(\lambda^*, \underset{\sim}{x}^*)^T \underset{\sim}{\psi}^* = 0 \qquad \underset{\sim}{e}_t^T \underset{\sim}{\psi}^* = 1\ .
\end{aligned}$$

Thus the operator

$$G(\lambda, \underset{\sim}{x}) - \Delta \underset{\sim}{e}_t \tag{5.4.2}$$

has a simple bifurcation point at $(\lambda^*, \underset{\sim}{x}^*)$ and we are dealing with the operator

$$G(\lambda, \underset{\sim}{x}) = \{G(\lambda, \underset{\sim}{x}) - \Delta \underset{\sim}{e}_t\} + \Delta \underset{\sim}{e}_t\ . \tag{5.4.3}$$

Having obtained (5.4.1) we can compute A, B and C of (4.1.5), either directly or using the approximations (3.5.20), and $D = \psi^{*T} \underset{\sim}{e}_t = 1$.

Now depending on the size of $|\Delta|$ we have three choices. If $|\Delta|$ is very small, e.g. the size of the round-off error, we may ignore it and regard $(\lambda^*, \underset{\sim}{x}^*)$ as a perfect bifurcation point. Then section 4.4 can be used to move along the solution-curves. For larger $|\Delta|$, and depending on the size of the $\delta\lambda$ or ε increment taken, we may incorporate a Δ factor into our starting values, as in Theorems 5.2, 5.5 and 5.8. If $|\Delta|$ is too large for suitable choice of $\delta\lambda$ or ε, our procedure depends on whether other singular points are present or not.

If C is non-zero and we are using continuation with respect to λ, then $\mathrm{sgn}(\Delta C)$ is the important variable. If $\Delta C < 0$ then both the perturbed solution curves cross the x-axis and we can set $\lambda = \lambda^*$ and compute either of these points using Theorem 5.3. We can then continue normally with λ. If $\Delta C > 0$ then each perturbed solution curve has a simple turning point with respect to λ and both can be computed using Theorem 5.4. We can then move along either of the branches emanating from the turning point by the method of section 3.6.

Exactly the same procedure is used if A is non-zero and ΔA positive or negative; except that in this case we compute points with $\varepsilon = 0$ using Theorem 5.6, or simple turning points with respect to ε using Theorem 5.7, and then continue with ε.

If A and C are both zero, then $\operatorname{sgn}(\Delta B)$ determines our plan. With $\Delta B < 0$ the two solution-curves both cross the line $\varepsilon = \delta\lambda$ while if $\Delta B > 0$ they cross the line $\varepsilon = -\delta\lambda$. In either case Theorem 5.9 allows us to compute nearby points on these curves. However, after having computed one of these points, we must be careful of which parameter is then used to move away along the solution curve. This is because the solution-curves of the perturbed problem consist of perturbed $\frac{1}{2}$ - curves of the non-perturbed problem, and one of these can only be parametrised by $\delta\lambda$ and the other only by ε. Thus we should continue with $\delta\lambda$ in one direction and ε in the other direction, according to the following table, where λ° is the λ-component of the solution computed from Theorem 5.9.

	$\lambda^\circ - \lambda^* > 0$	$\lambda^\circ - \lambda^* < 0$
$\Delta B > 0$	$\delta\lambda > 0$ $\varepsilon < 0$	$\delta\lambda < 0$ $\varepsilon > 0$
$\Delta B < 0$	$\varepsilon > 0$ $\delta\lambda > 0$	$\varepsilon < 0$ $\delta\lambda < 0$

Fig. 5.3

5.5 Numerical Results

Our examples come from the class of equations defined in section 4.5 but now we chose different functions $V(t)$ in (4.5.5a). The choice there meant that perfect bifurcation was retained after discretization because our finite-difference approximation is exact upto cubics.

Example 1

We take $V(t) = t(1-t)\exp(t)$ and $h = 1/32$. Table 1 shows the convergence to the imperfect bifurcation point.

Table 1

λ	$\|\underset{\sim}{x}\|_2$	$\|\delta\lambda\|$	$\|\delta\underset{\sim}{x}\|_2$	$\|\delta\underset{\sim}{\phi}\|_2$
1.001118066	1.018515803			
		0.26 E-2	0.52 E-1	0.34 E-2
0.9985090522	1.069035073			
		0.35 E-3	0.16 E-3	0.57 E-4
0.9988568088	1.068964652			
		0.13 E-6	0.14 E-6	0.19 E-7
0.9988569353	1.068986793			
		0.12 E-14	0.60 E-14	0.91 E-15
0.9988569353	1.068964793			

The values of A, B, C, D and Δ at the imperfect bifurcation point are

a) $A = -0.10$ E-1 b) $B = 0.32$ E-2

c) $C = 0.20$ E-2 d) $D = 0.25$ (5.5.1)

e) $\Delta = -0.20$ E-4

For this example the solution curves in the neighbourhood of the imperfect bifurcation point were computed as for perfect bifurcation, but with starting-values modified to include a Δ -term. $\delta\lambda$ was taken to be 0.05 and the results are given in Table 2.

Table 2a : Perturbed Primary Curve

	$\|\|x\|\|_2$		$\|\|x\|\|_2$
$\delta\lambda = -0.05$	0.9788997775	$\delta\lambda = 0.05$	1.159033307
	0.9792577417		1.159319640
	0.9792581779		1.159319333
	0.9792581779		1.159319333

Table 2b : Perturbed Bifurcating Curve

	$\|\|x\|\|_2$		$\|\|x\|\|_2$
$\delta\lambda = -0.05$	1.230989106	$\delta\lambda = 0.05$	0.9091081558
	1.244211275		0.9197494264
	1.243550087		0.9202292451
	1.243548348		0.9202302142
	1.243548348		0.9202302142

Example 2

We again choose $V(t) = t(1-t)\exp(t)$ but this time with $h = 1/12$. The convergence to the imperfect bifurcation point is shown in Table 3.

Table 3

λ	$\|\underset{\sim}{x}\|_2$	$\|\delta\lambda\|$	$\|\delta\underset{\sim}{x}\|_2$	$\|\delta\underset{\sim}{\phi}\|_2$
0.9996812334	0.5677721432			
		0.11 E-1	0.85 E-1	0.61 E-2
0.9889976358	0.6506869975			
		0.29 E-2	0.71 E-3	0.30 E-3
0.9919090932	0.6505294821			
		0.79 E-5	0.59 E-5	0.75 E-6
0.9919170361	0.6505352767			
		0.32 E-11	0.17 E-10	0.18 E-11
0.9919170361	0.6505352767			

The values of the important constants at the imperfect bifurcation point are

a) $A = -0.44$ E-1 b) $B = 0.23$ E-1

c) $C = 0.24$ E-1 d) $D = 0.41$ (5.5.2)

e) $\Delta = -0.38$ E-3

Because $\Delta CD < 0$ the solution curves cross the line $\lambda = \lambda^*$ and these two points are calculated in Table 4.

Table 4

$\|\underset{\sim}{x}\|_2$	$\|\underset{\sim}{x}\|_2$
0.5715033389	0.7299422333
0.5715601664	0.7299972486
0.5715606072	0.7299967946
0.5715606072	0.7299967946

Example 3

We again take $h \doteq 1/12$ but this time with $V(t) = t(1-t)\cos(t)$. Table 5 shows the computation of the imperfect bifurcation point.

Table 5

λ	$\|\underset{\sim}{x}\|_2$	$\|\delta\lambda\|$	$\|\delta\underset{\sim}{x}\|_2$	$\|\delta\underset{\sim}{\phi}\|_2$
0.9548653267	0.3495150408			
		0.39 E-1	0.22 E-1	0.24 E-2
0.9940647300	0.3283133584			
		0.14 E-2	0.34 E-3	0.95 E-4
0.9954829204	0.3286037934			
		0.28 E-6	0.33 E-6	0.56 E-7
0.9954831962	0.3286041107			
		0.27 E-13	0.16 E-13	0.11 E-13
0.9954831962	0.3286041107			

The values of A, B, C, D and Δ are

a) $A = -0.28$ E-1 b) $B = 0.45$ E-1

c) $C = 0.24$ E-1 d) $D = 0.41$ (5.5.3)

e) $\Delta = 0.19$ E-3

and thus $\Delta CD > 0$ and there is a simple turning point on each solution curve. These are computed in Table 6.

Table 6 a

λ	$\lVert \underline{x} \rVert_2$	$\lvert \delta\lambda \rvert$	$\lVert \delta\underline{x} \rVert_2$	$\lVert \delta\underline{\phi} \rVert_2$
0.9578306412	0.3635154226			
		0.76 E-3	0.15 E-2	0.35 E-2
0.9570643827	0.3670525764			
		0.24 E-4	0.26 E-4	0.28 E-5
0.9570408028	0.3670787005			
		0.77 E-8	0.73 E-8	0.69 E-9
0.9570408105	0.3670786933			
		0.68 E-15	0.94 E-15	0.79 E-16
0.9570408105	0.3670786933			

Table 6 b

λ	$\lVert \underline{x} \rVert_2$	$\lvert \delta\lambda \rvert$	$\lVert \delta\underline{x} \rVert_2$	$\lVert \delta\underline{\phi} \rVert_2$
1.034662201	0.2948934704			
		0.73 E-3	0.16 E-2	0.35 E-2
1.035395168	0.2933292895			
		0.25 E-4	0.24 E-4	0.29 E-5
1.035420487	0.2933057173			
		0.75 E-8	0.72 E-8	0.68 E-9
1.035420480	0.2933057244			
		0.73 E-15	0.68 E-15	0.52 E-16
1.035420480	0.2933057244			

The values of the important constants $\underline{\phi}^{*T} G_\lambda(\lambda^*, \underline{x}^*)$ and $\underline{\phi}^{*T} G_{xx}(\lambda^*, \underline{x}^*)\underline{\phi}^*\underline{\phi}^*$ at these two points are respectively

a) 0.40 E-2 and 0.45 E-1

b) -0.37 E-2 and 0.49 E-1 . (5.5.4)

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